Functions of two or more variables

While we have had great success using calculus to find maxima and minima of functions of one variable, the truth is that most phenomena are dependent on more than one variable. For example the number of lemons on a lemon tree depends on

the volume of water given	x l/year
the number of parasites	y
the amount of fertiliser	z
the severity of frosts	t
etc	$\ldots$

n is a function of many variables: $n = f(x, y, z, t, \ldots)$.

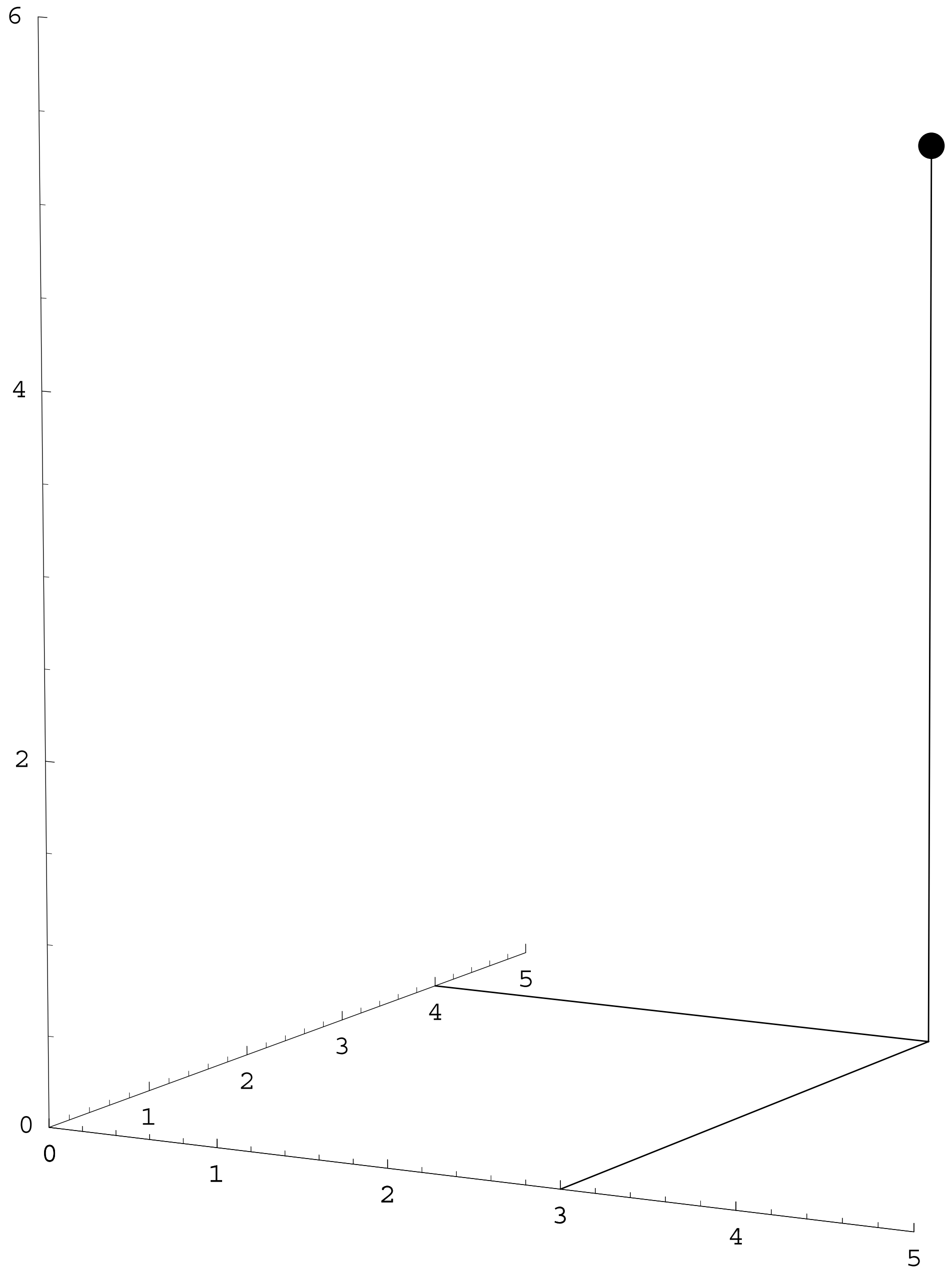
Functions of two variables have recipes like:

$$f(x,y) = x + xy + y^2$$

$$\begin{aligned} f(0,0) &= 0 + 0 \times 0 + 0^2 = 0 \\ f(1,0) &= 1 + 1 \times 0 + 0^2 = 1 \\ f(1,1) &= 1 + 1 \times 1 + 1^2 = 3 \\ f(-1,2) &= -1 + (-1) \times 2 + 2^2 = 1 \end{aligned}$$

Finding maxima and minima of functions of two or more variables requires a few new ideas..

We shall look at functions of two variables because the passage to more than two variables doesn't need new ideas and because we can (just about) draw graphs for functions of two variables: they look like surfaces in three-dimensional space.


6
4
2
0
0
1
2
3
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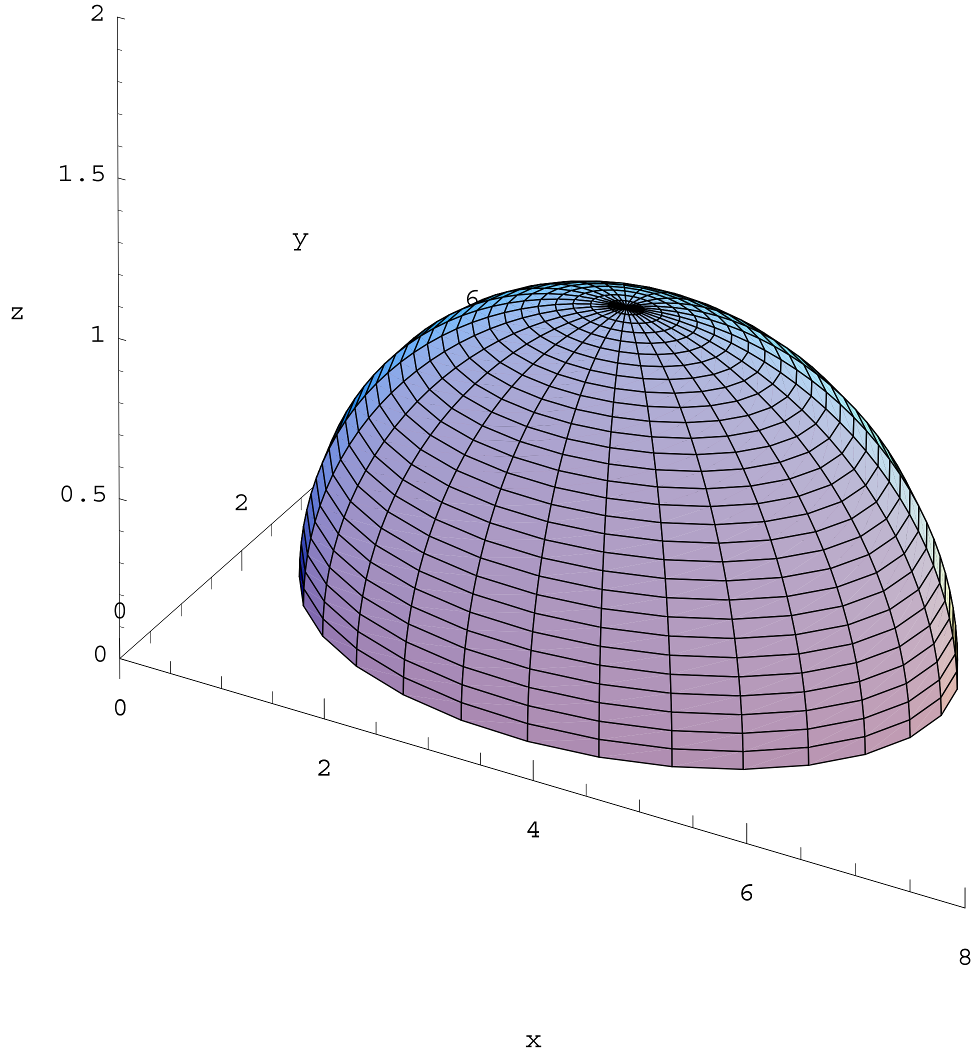
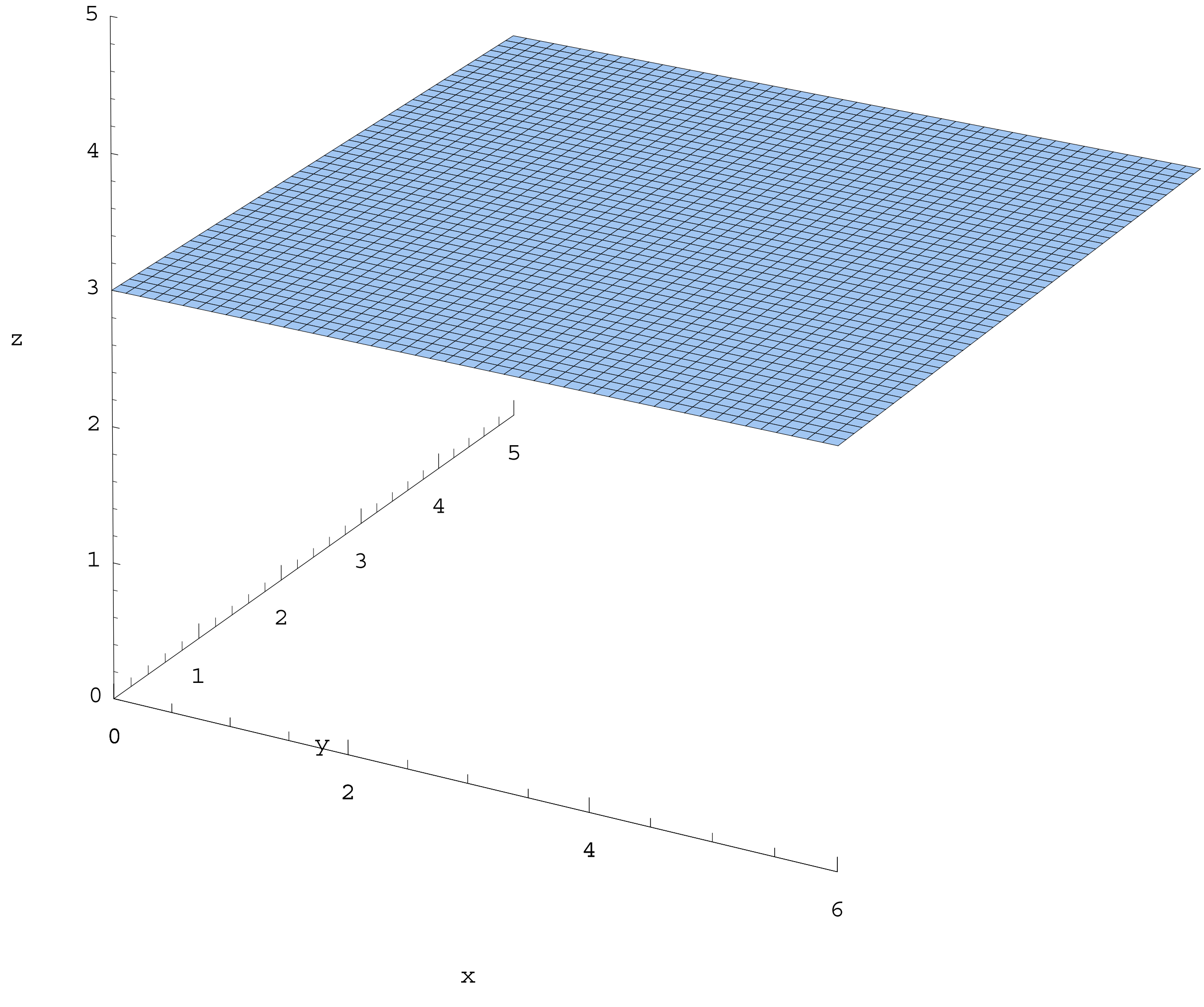
5
1
2
3
4
5

As $(x, y, 0)$ runs over a region in the xy-plane $f\,(x, y, f(x, y))$ runs over a surface in 3D space.

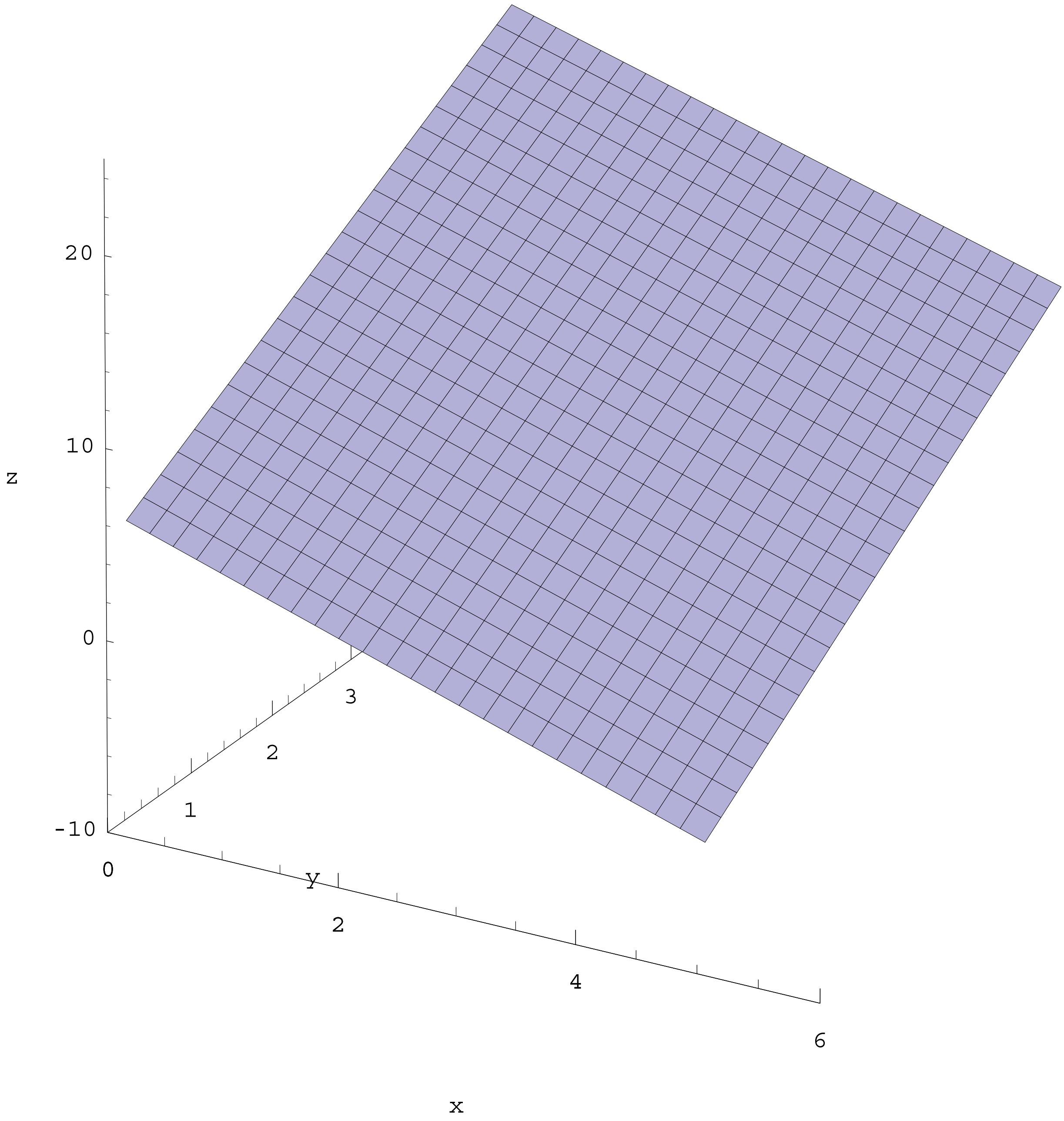


We now look at some of these "graphs."

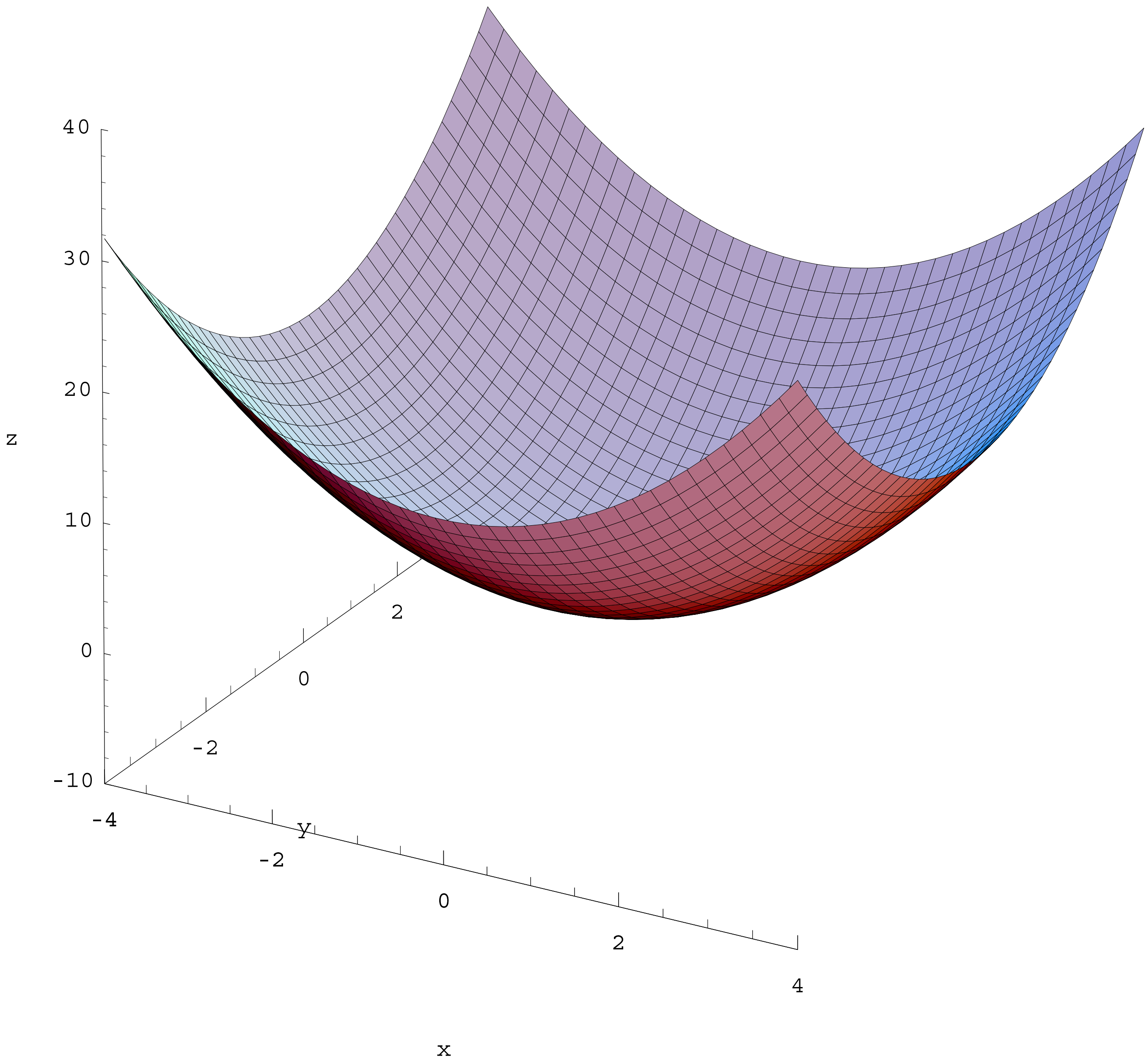
$$f(x, y) = 3$$



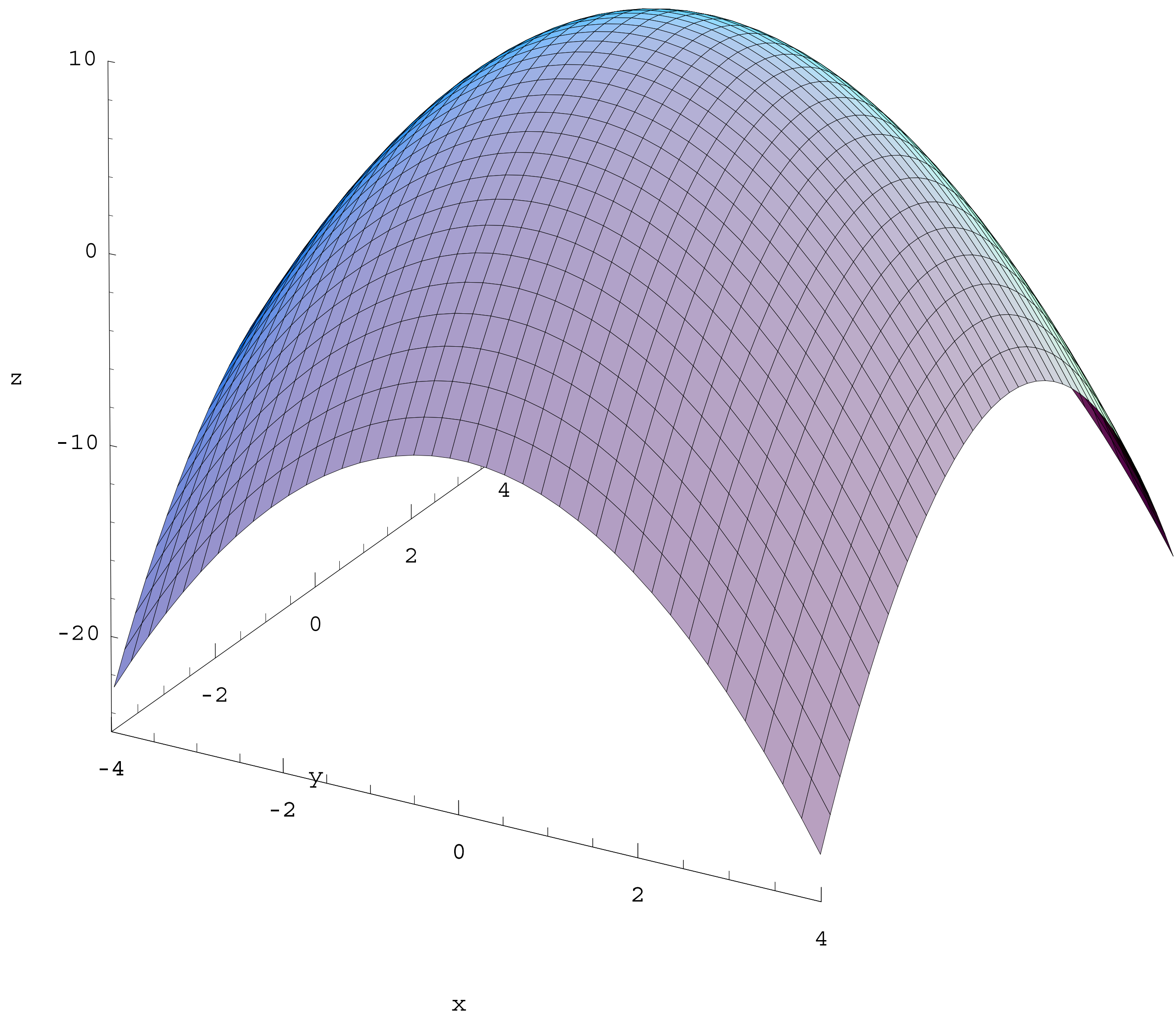
$$f(x, y) = -2x + 3y + 6$$



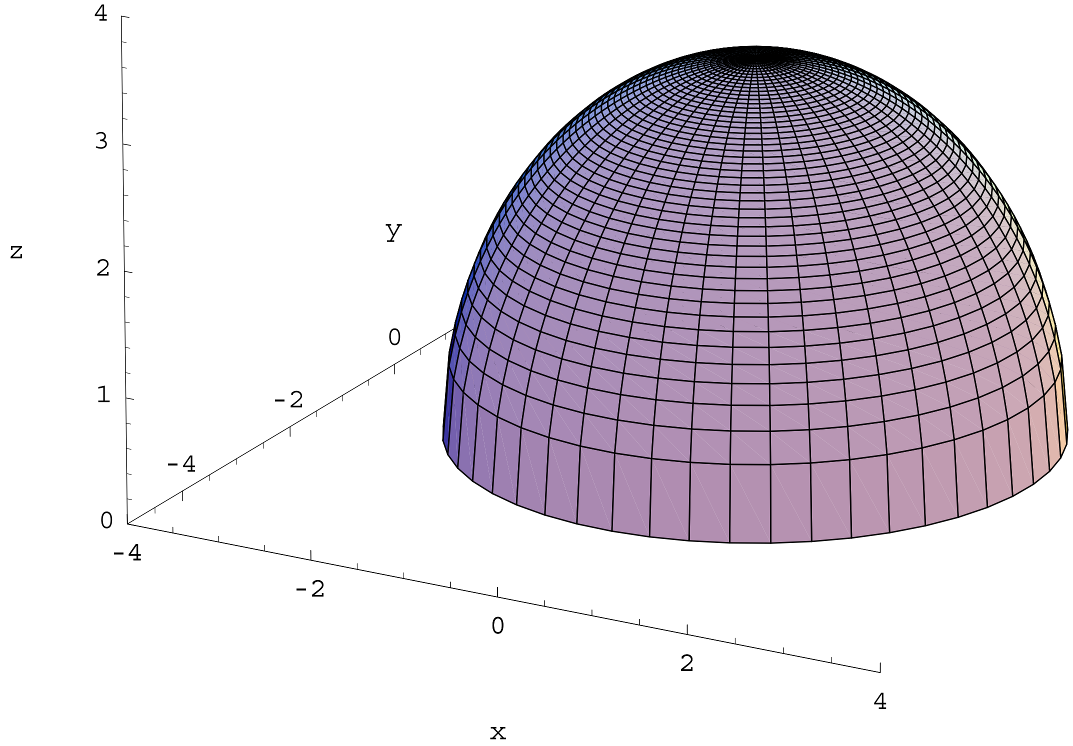
$$f(x,y) = x^2 + y^2$$



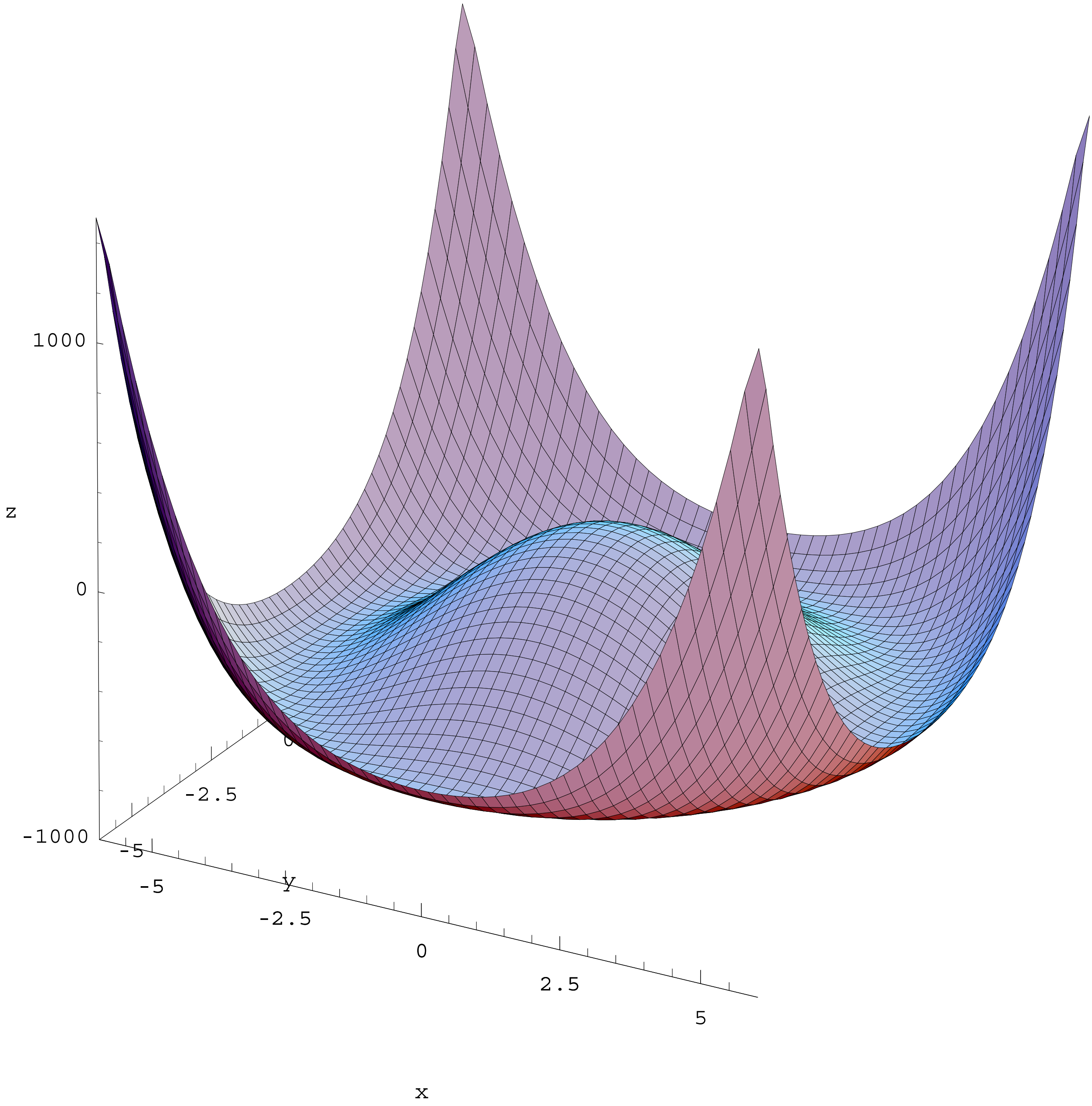
$$f(x, y) = 9 - x^2 - y^2$$



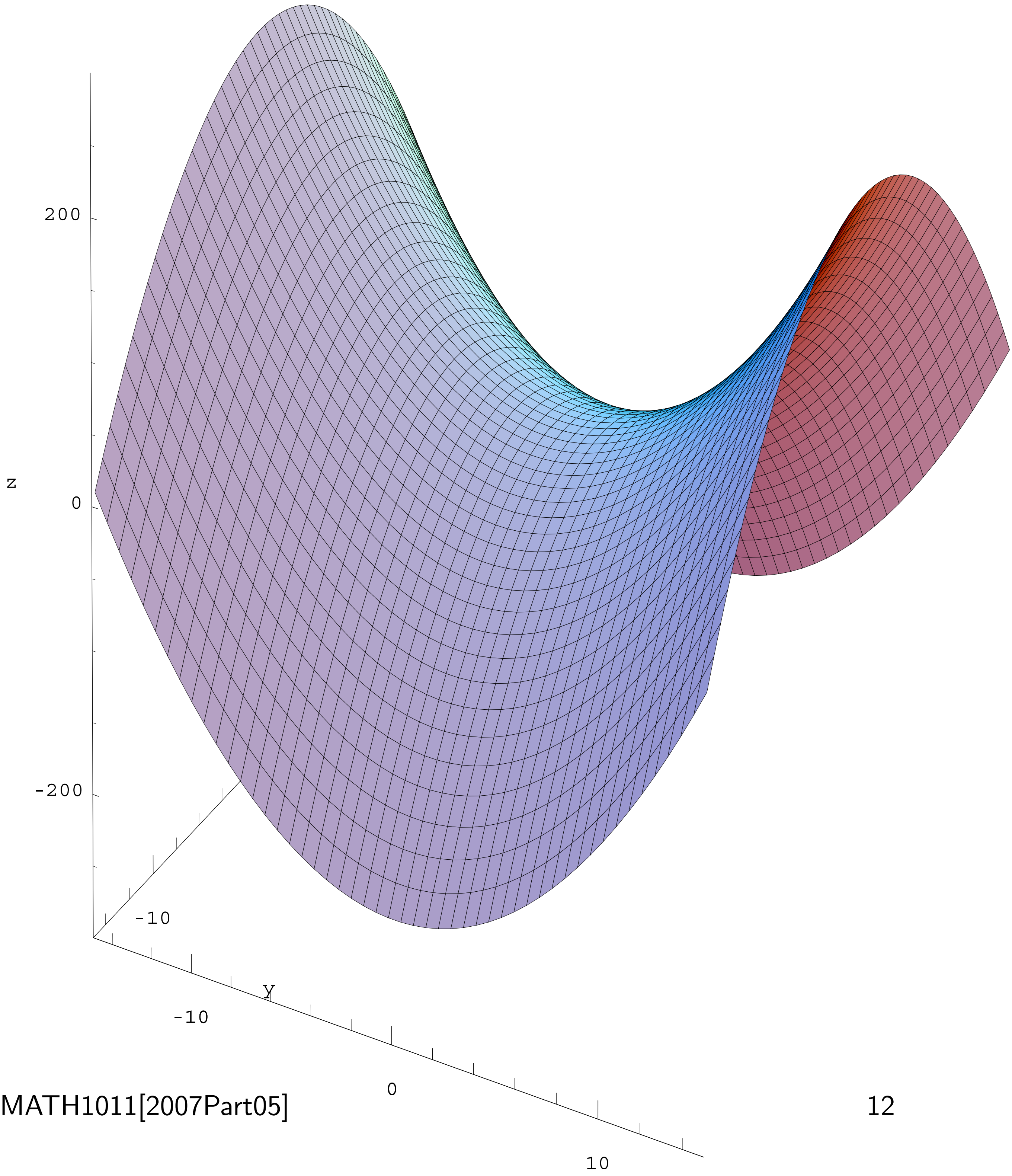
$$f(x,y) = \sqrt{9 - x^2 - y^2}$$



$$f(x,y) = (x^2+y^2)^2 - 50(x^2+y^2)$$



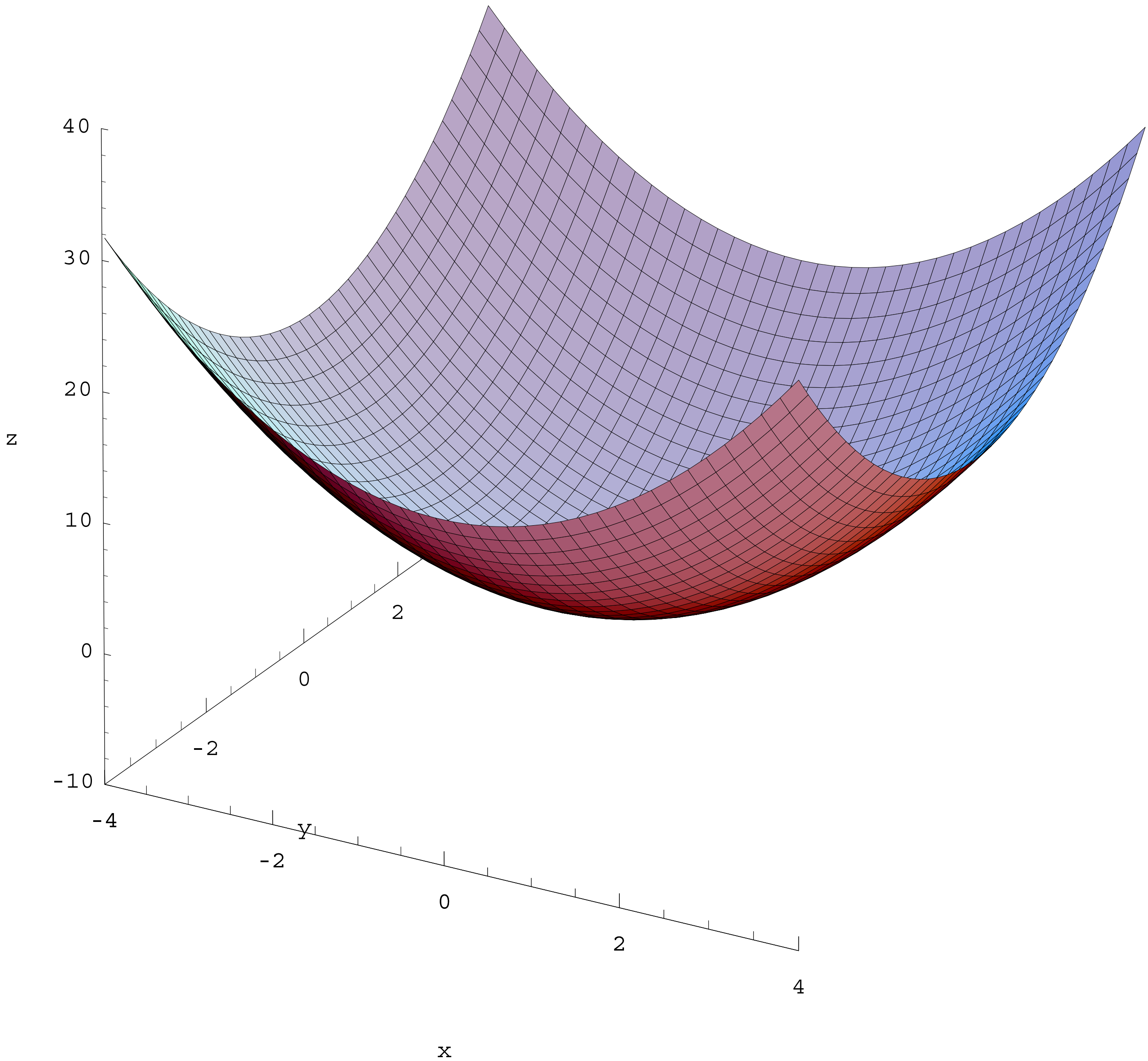
$$f(x, y) = x^2 - y^2$$



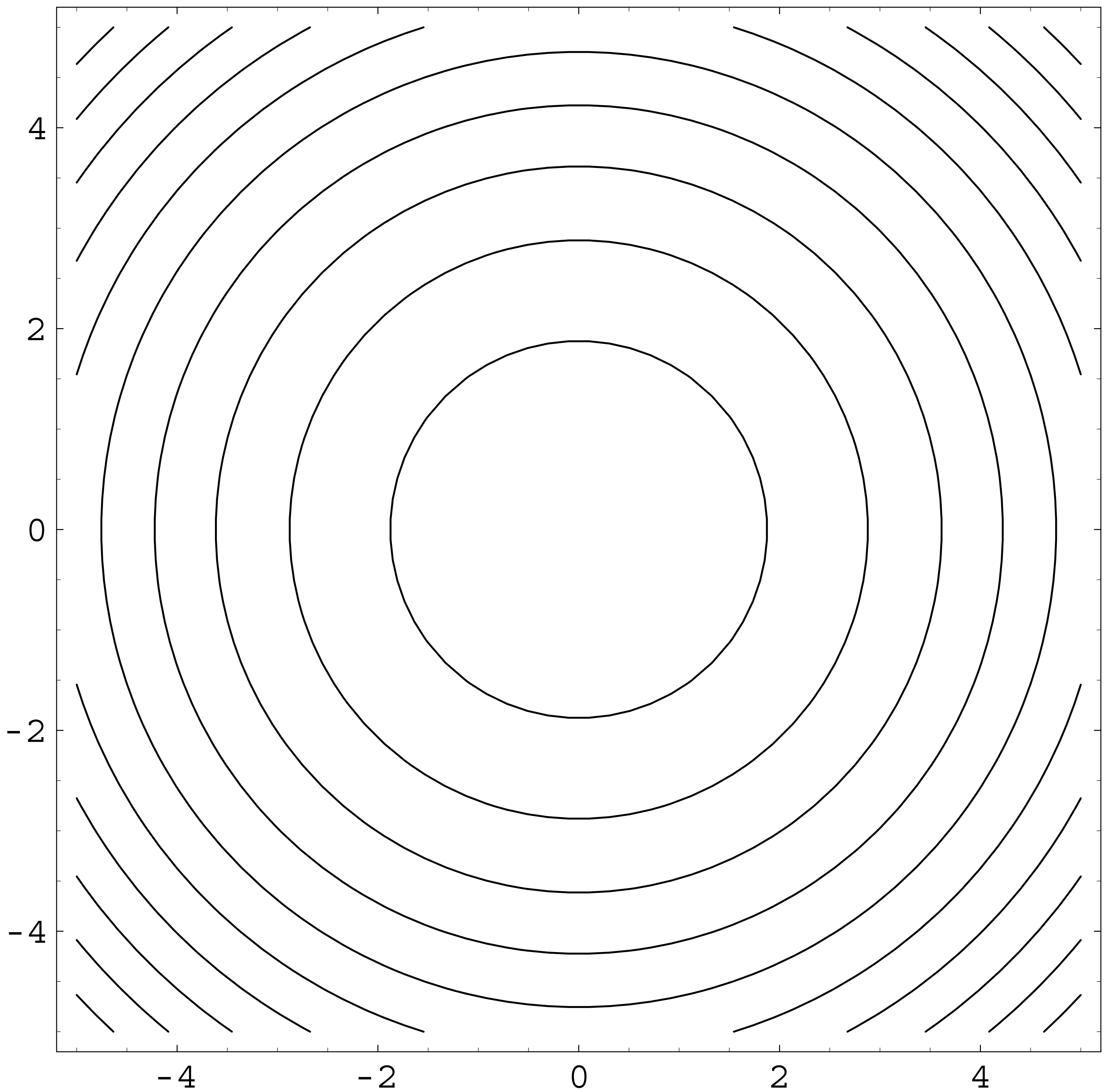
A few ideas which make it easier to understand these graphs are:

- The graphs above where $f(x, y) = g\left((x^2 + y^2\right)$ are surfaces of revolution: you revolve $z = g(x)$ about the $x$axis.
- You can draw "contour plots," like contours on a topographical map (horizontal sections).
- You can think about other sorts of section.

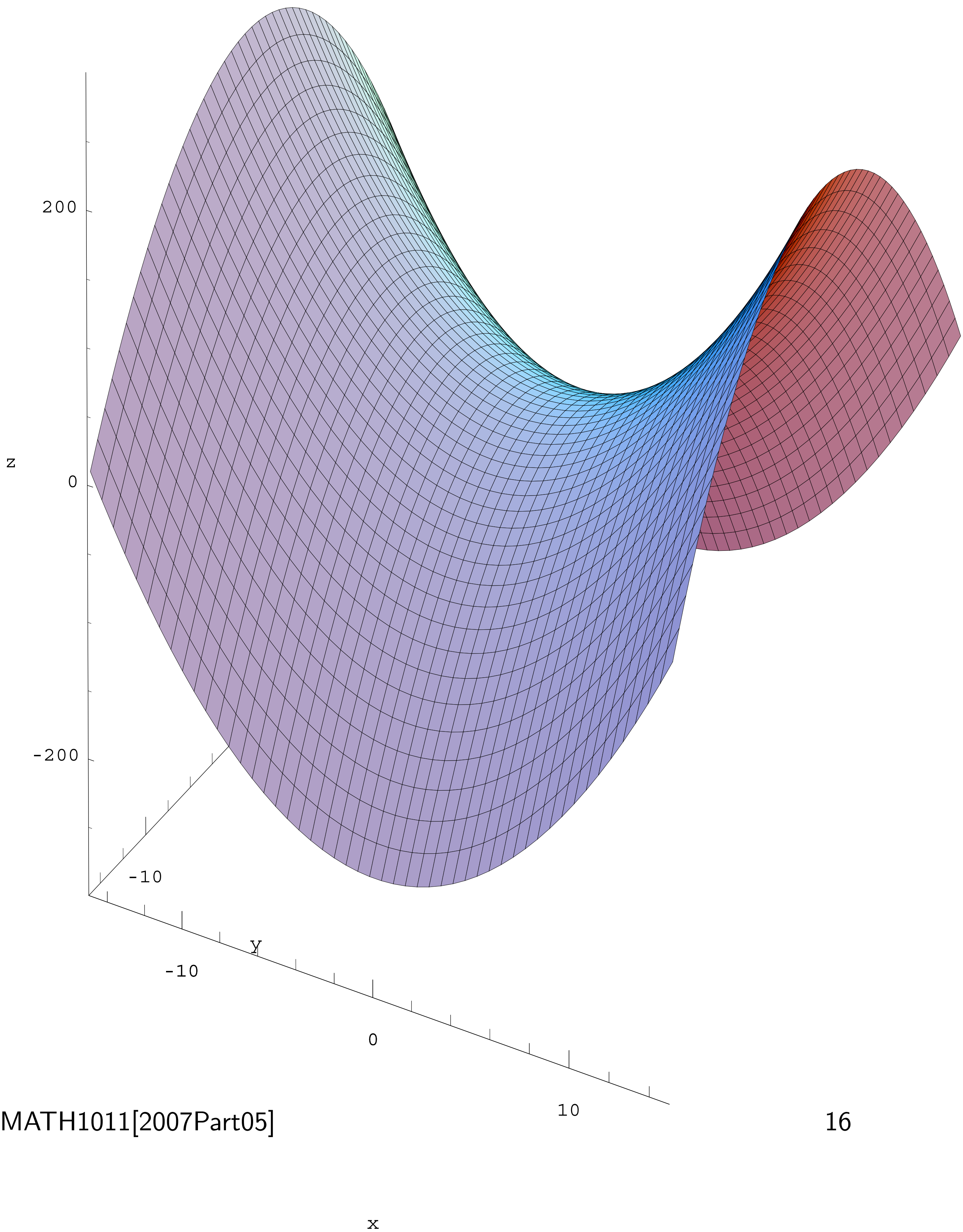
Graph of $z = x^2 + y^2$



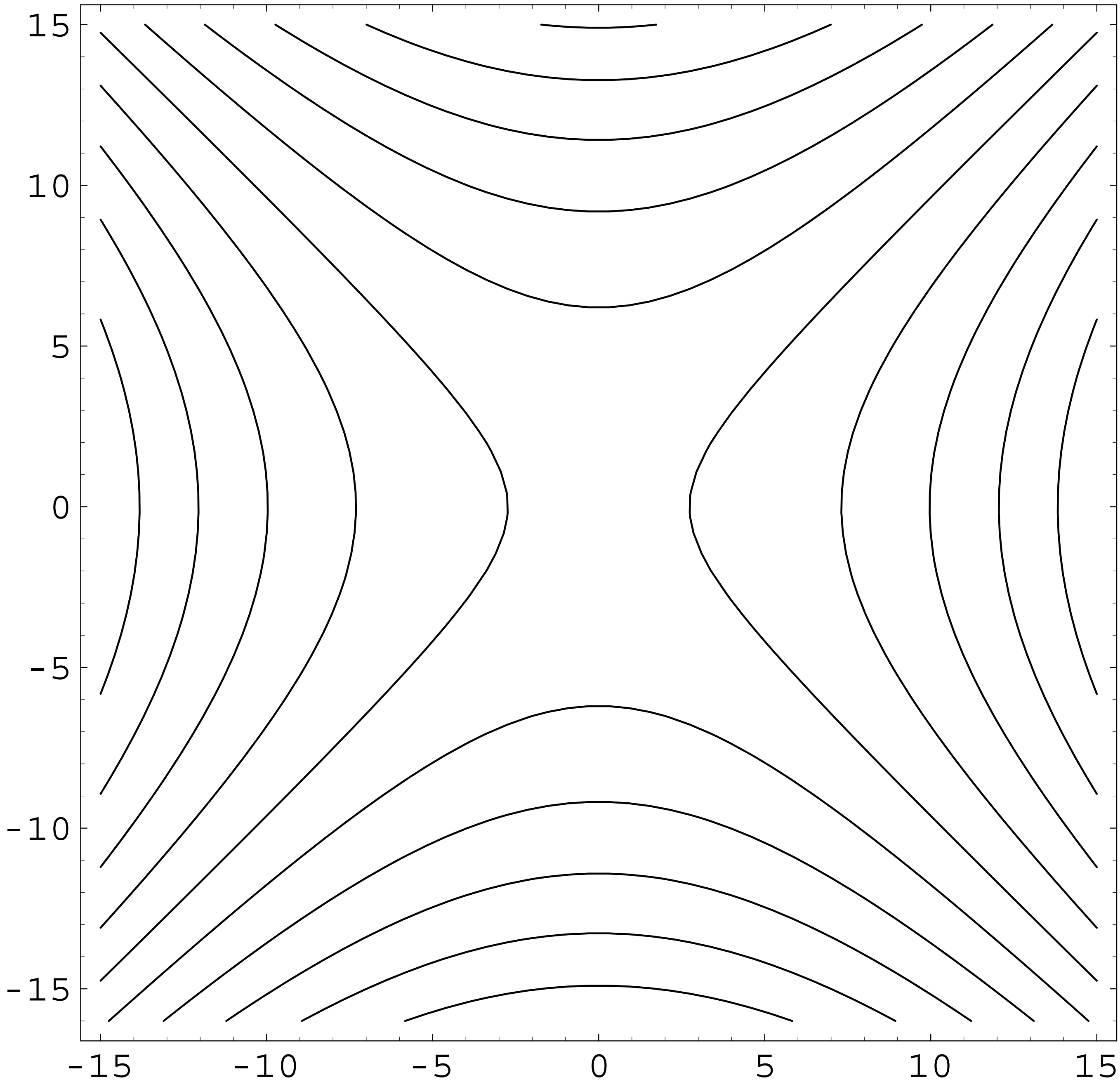
Contour Map of $z = x^2 + y^2$



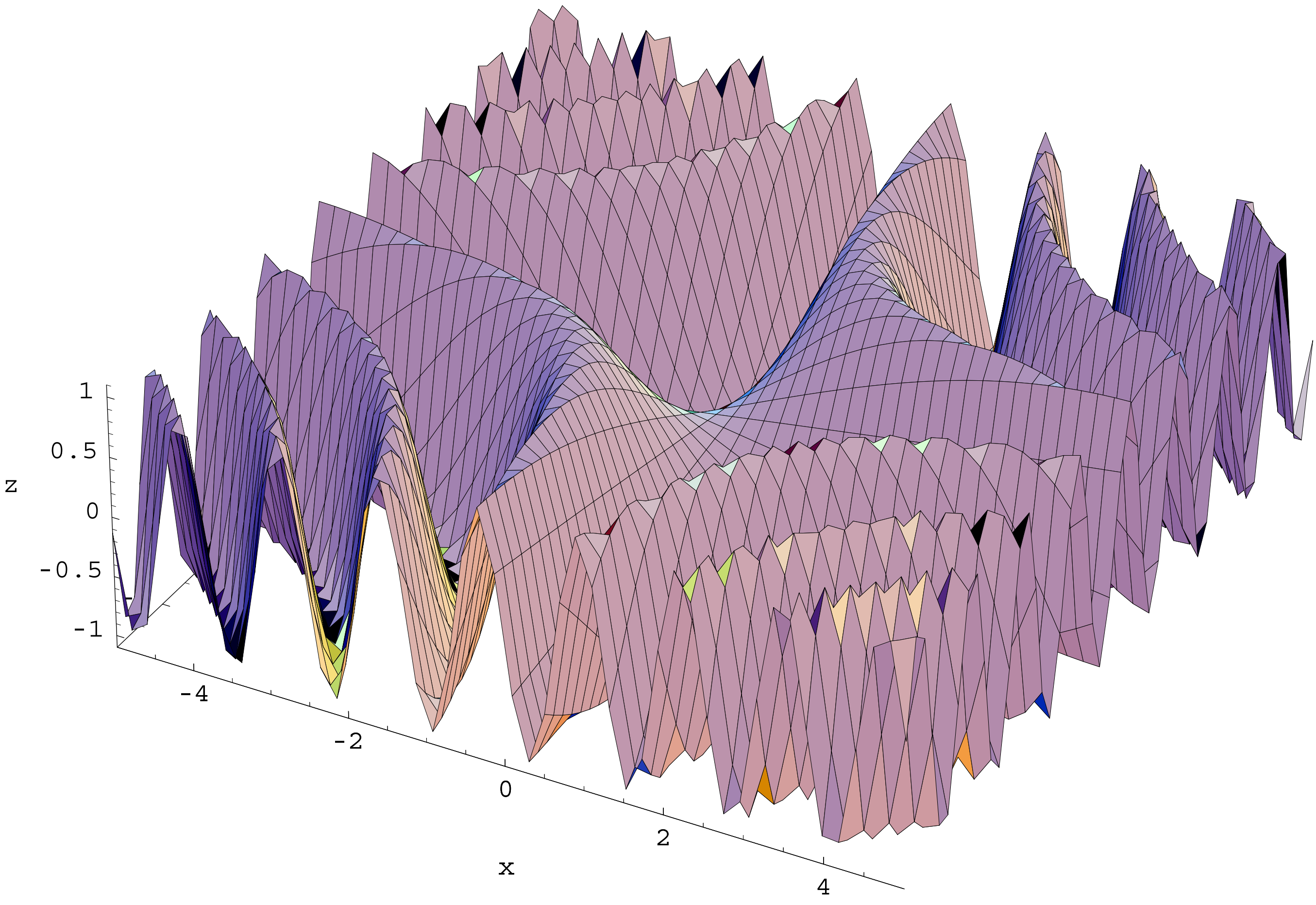
Graph of $z = x^2 - y^2$



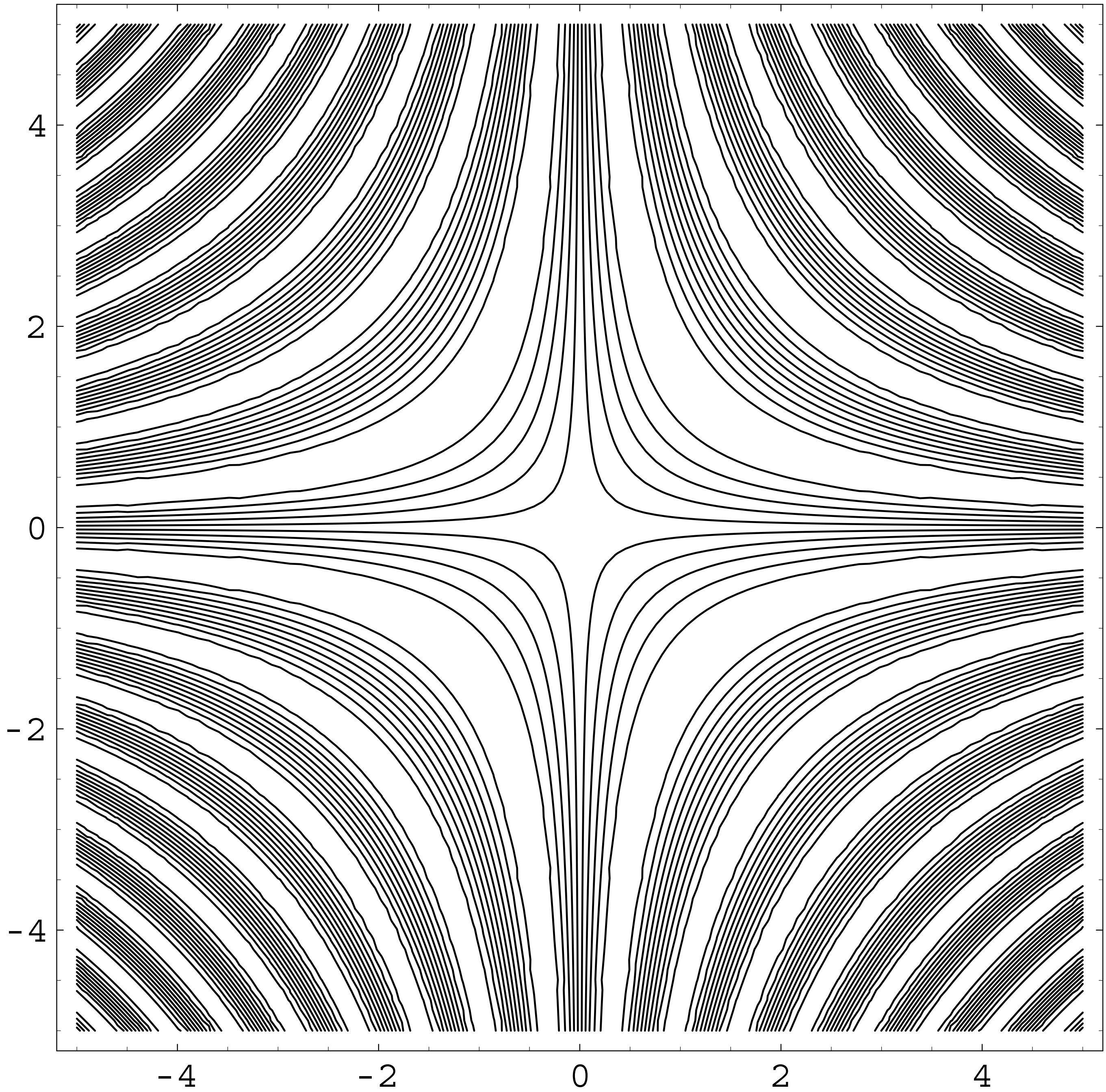
Contour Map of $z = x^2 - y^2$



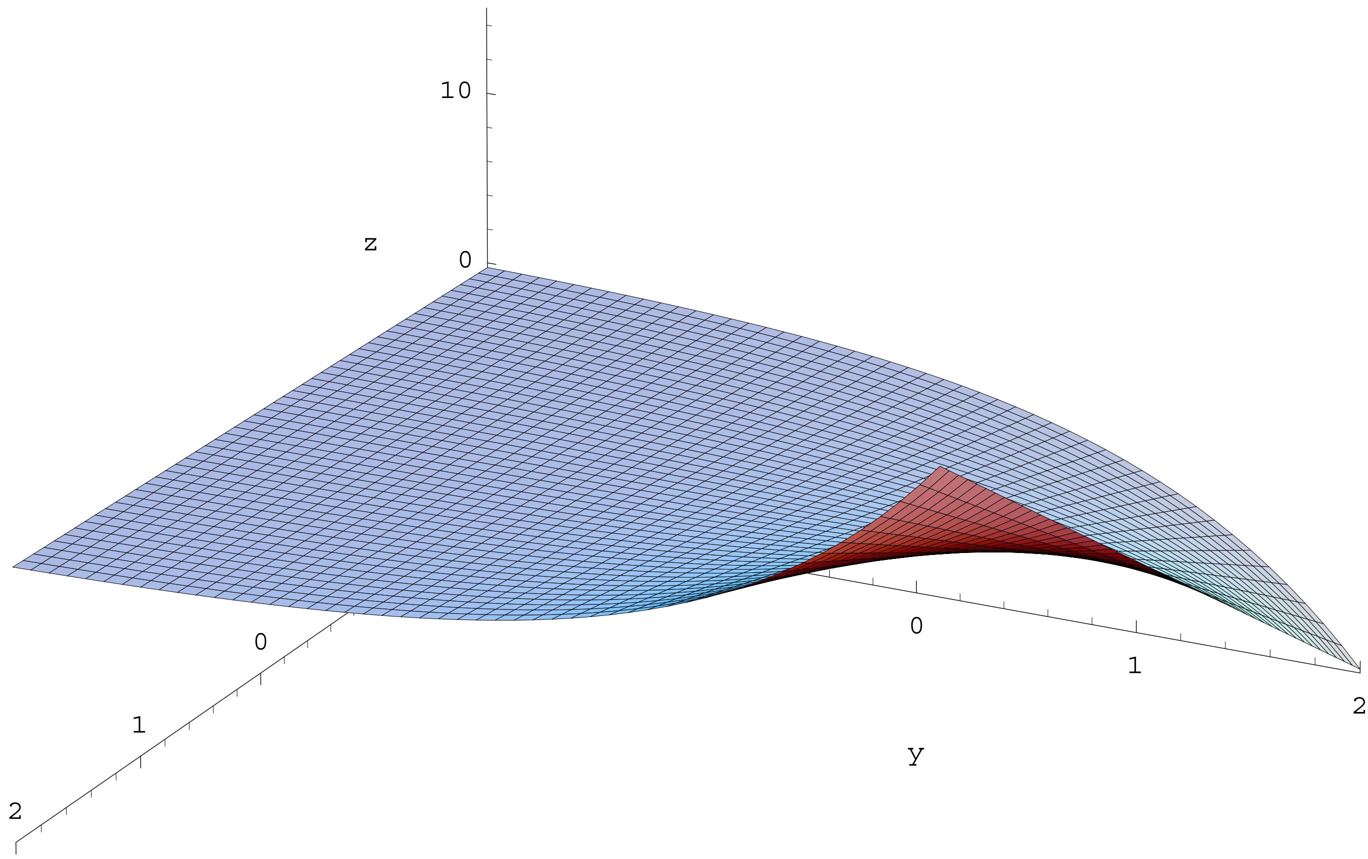
Graph of $z = \sin xy$



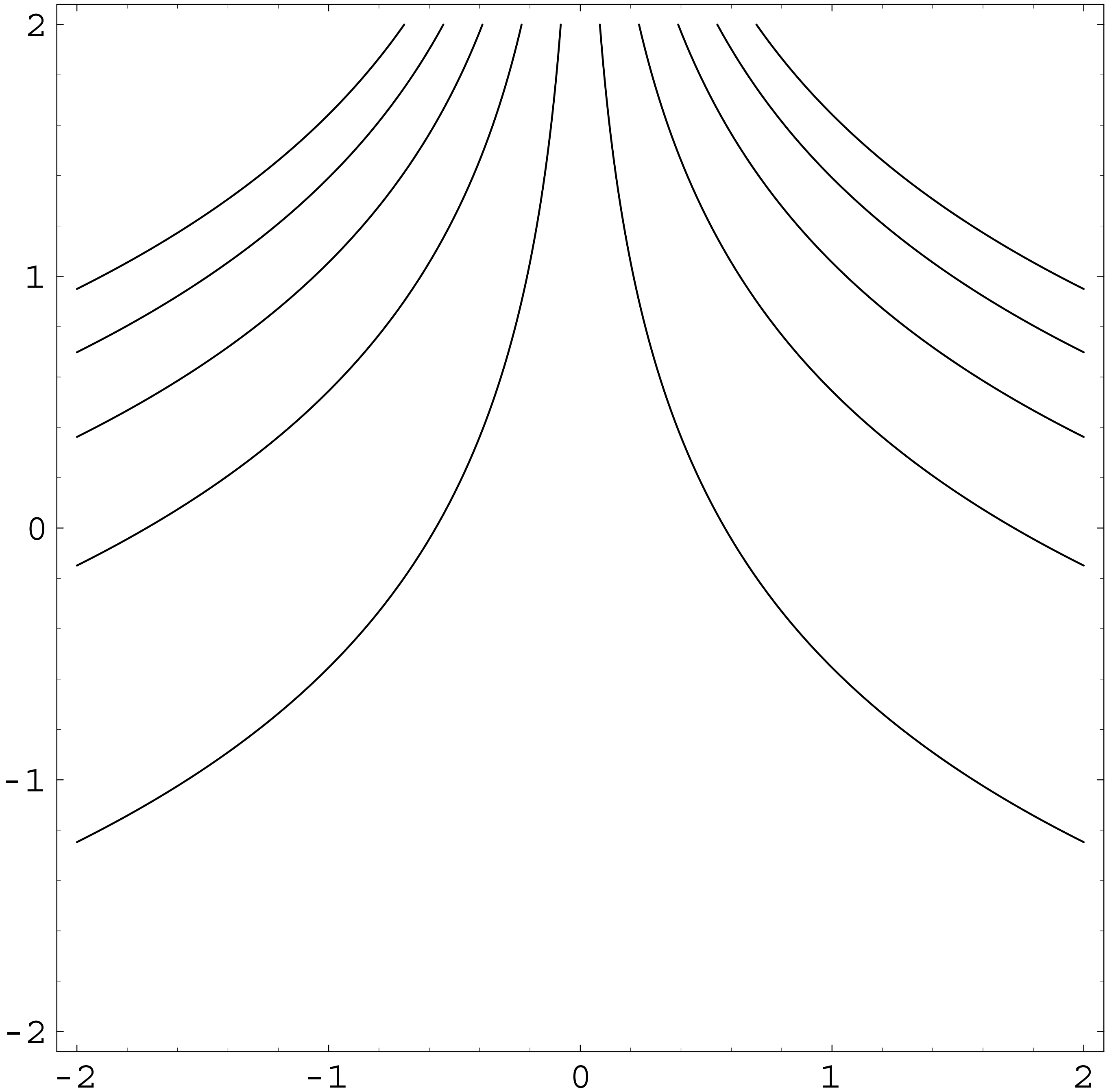
CONTOUR MAP OF $z = \sin xy$



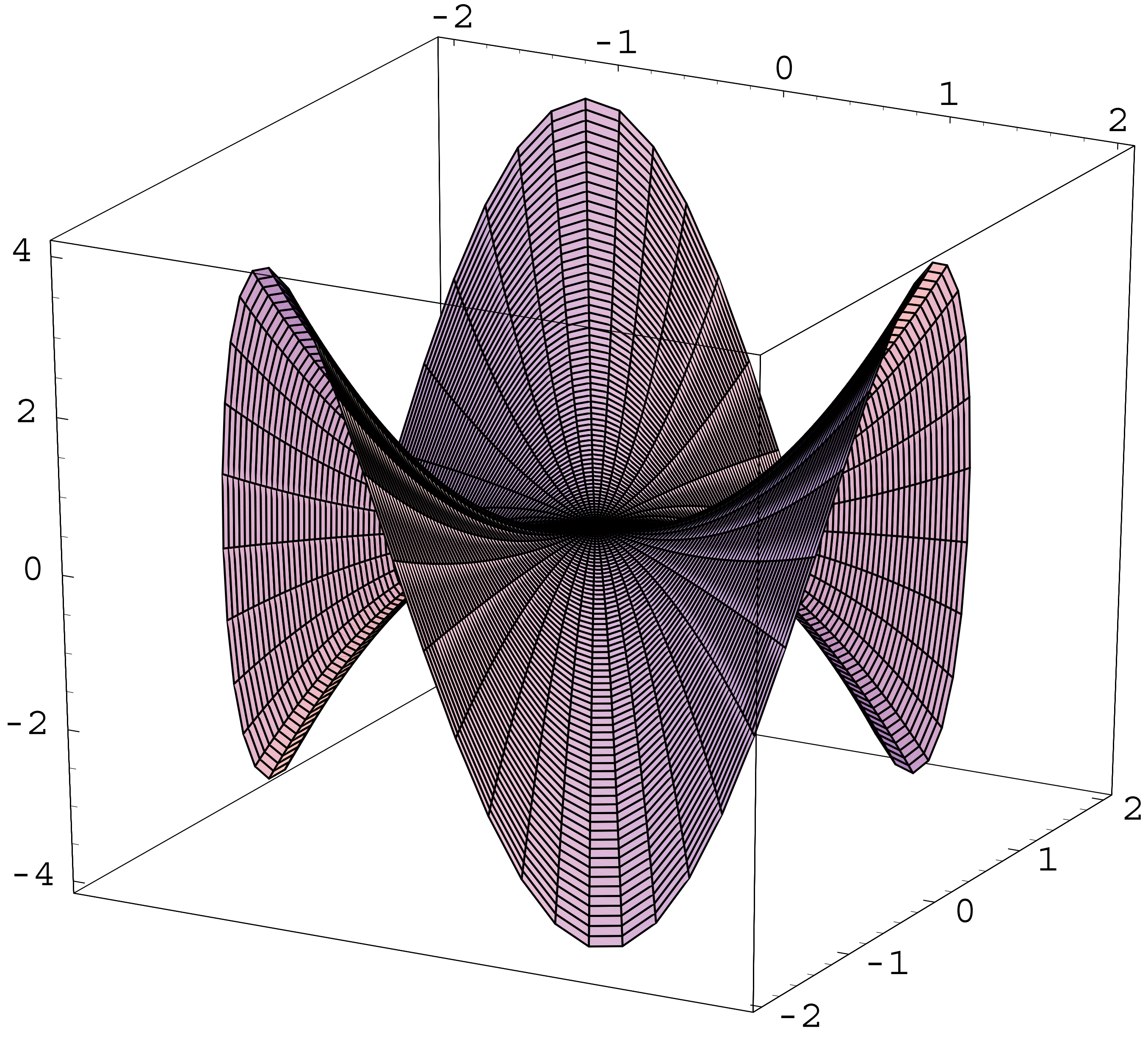
Graph of $z = xe^y$



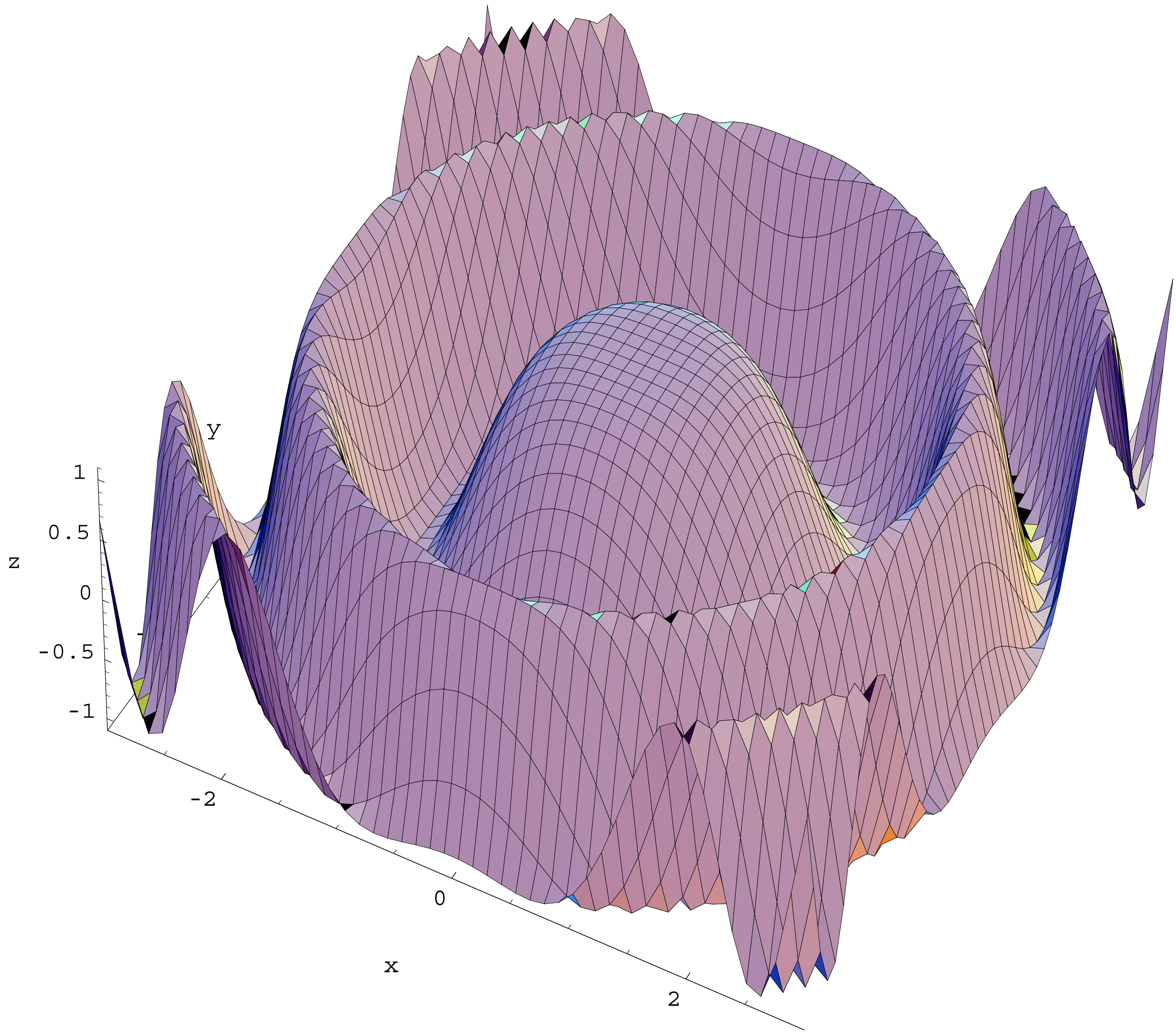
Contour Map of $z = xe^y$



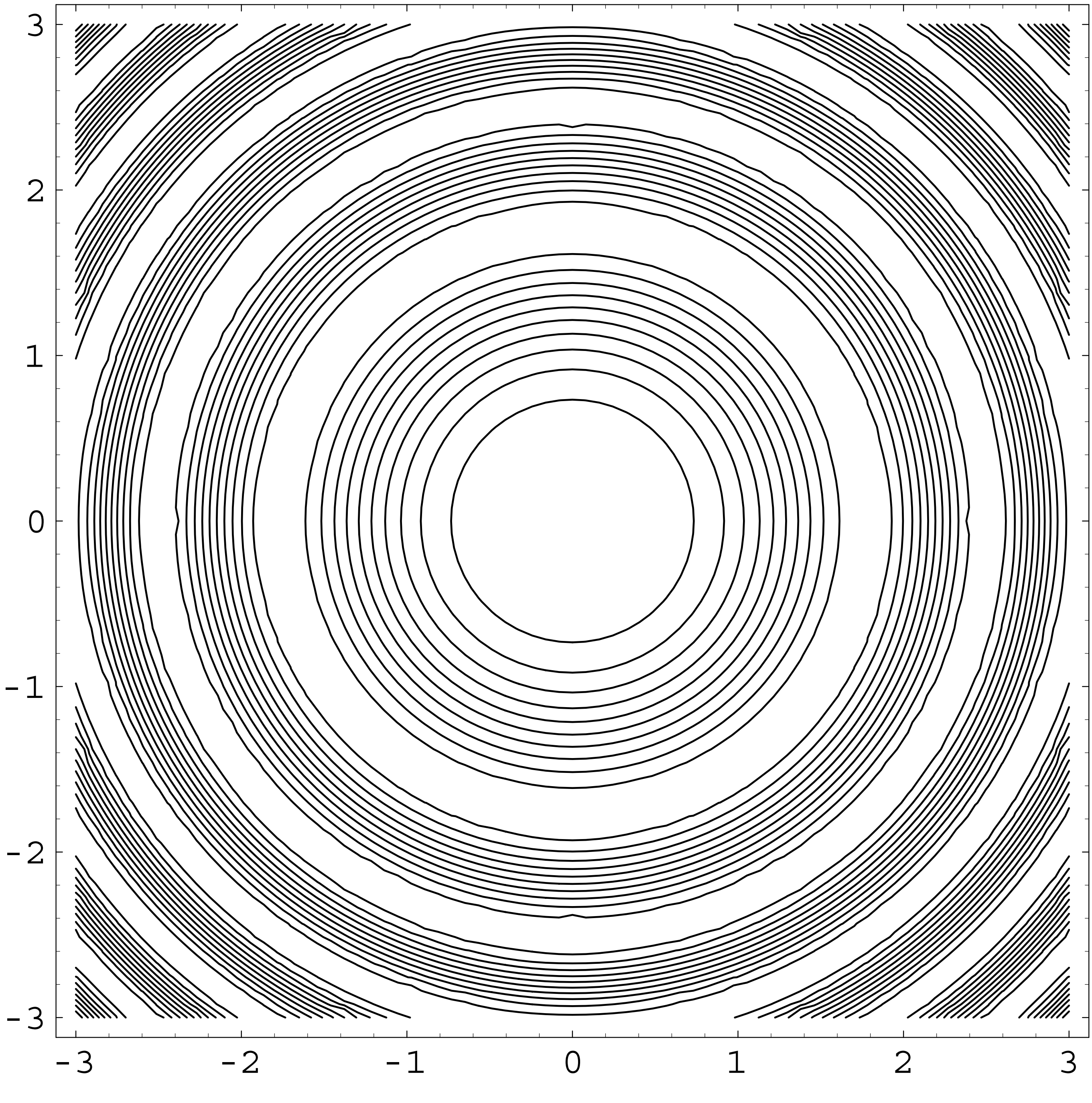
Monkey Saddle



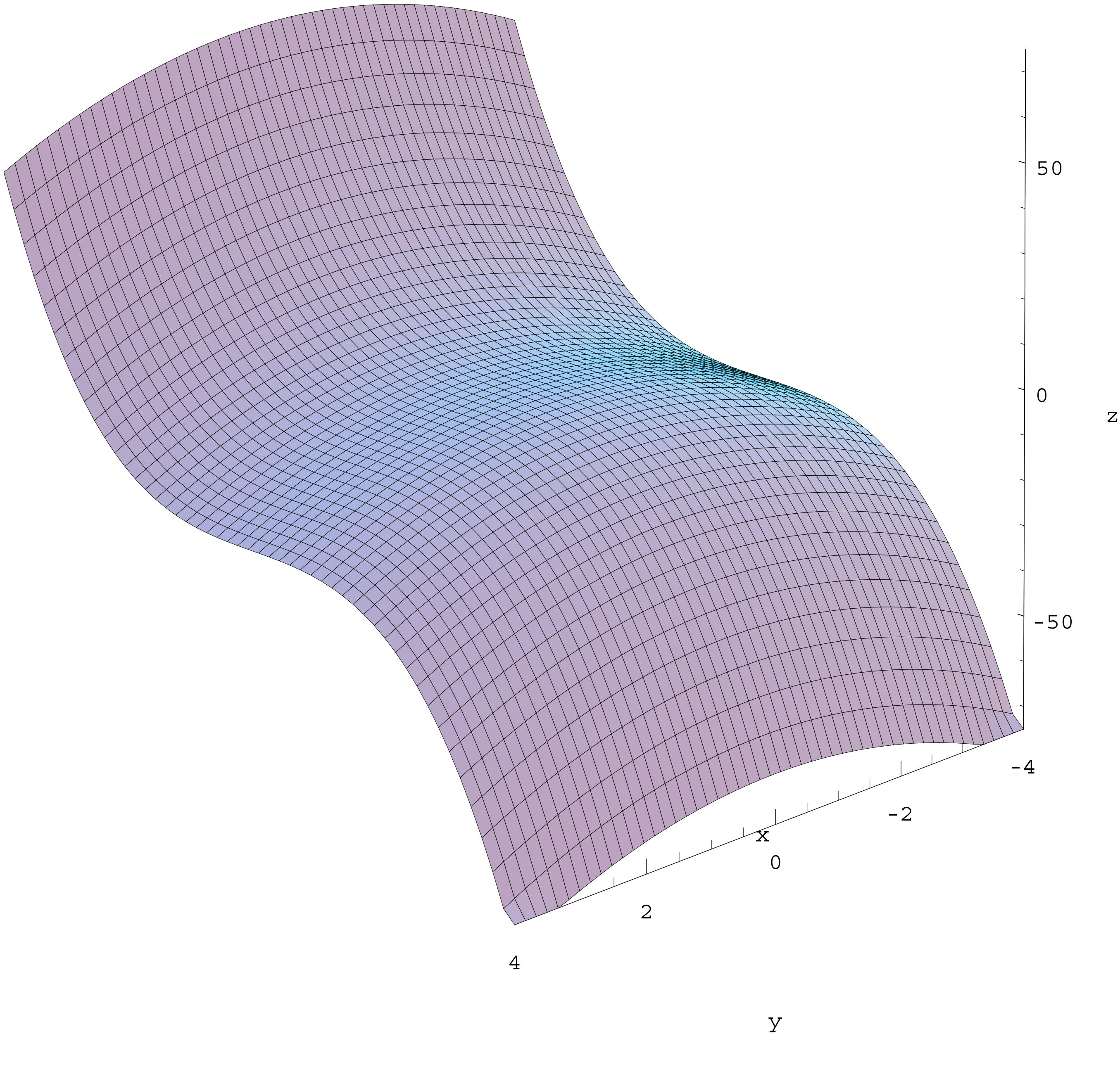
Graph of $z = \cos(x^2 + y^2)$



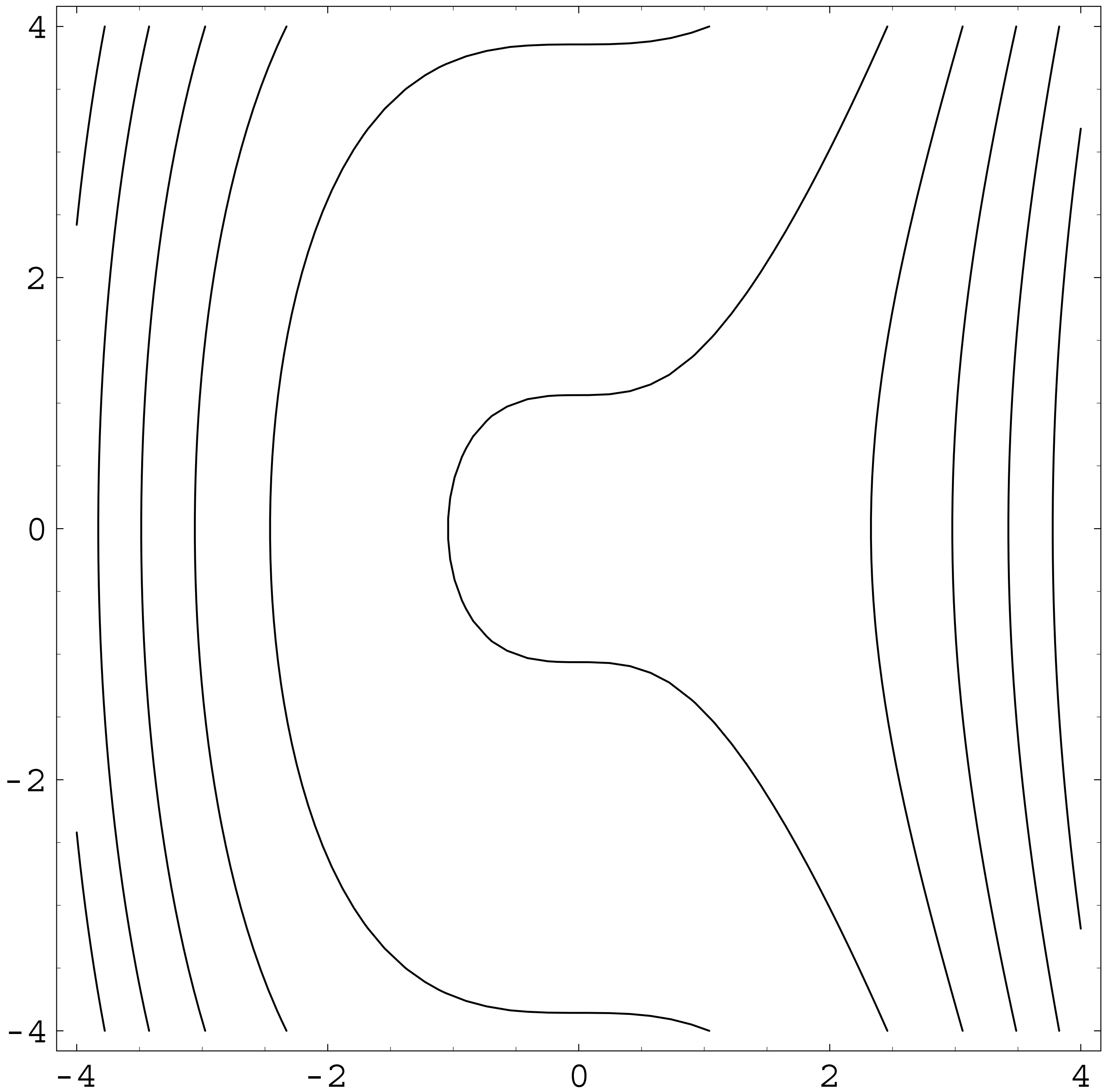
Contour Map of $z = \cos(x^2 + y^2)$



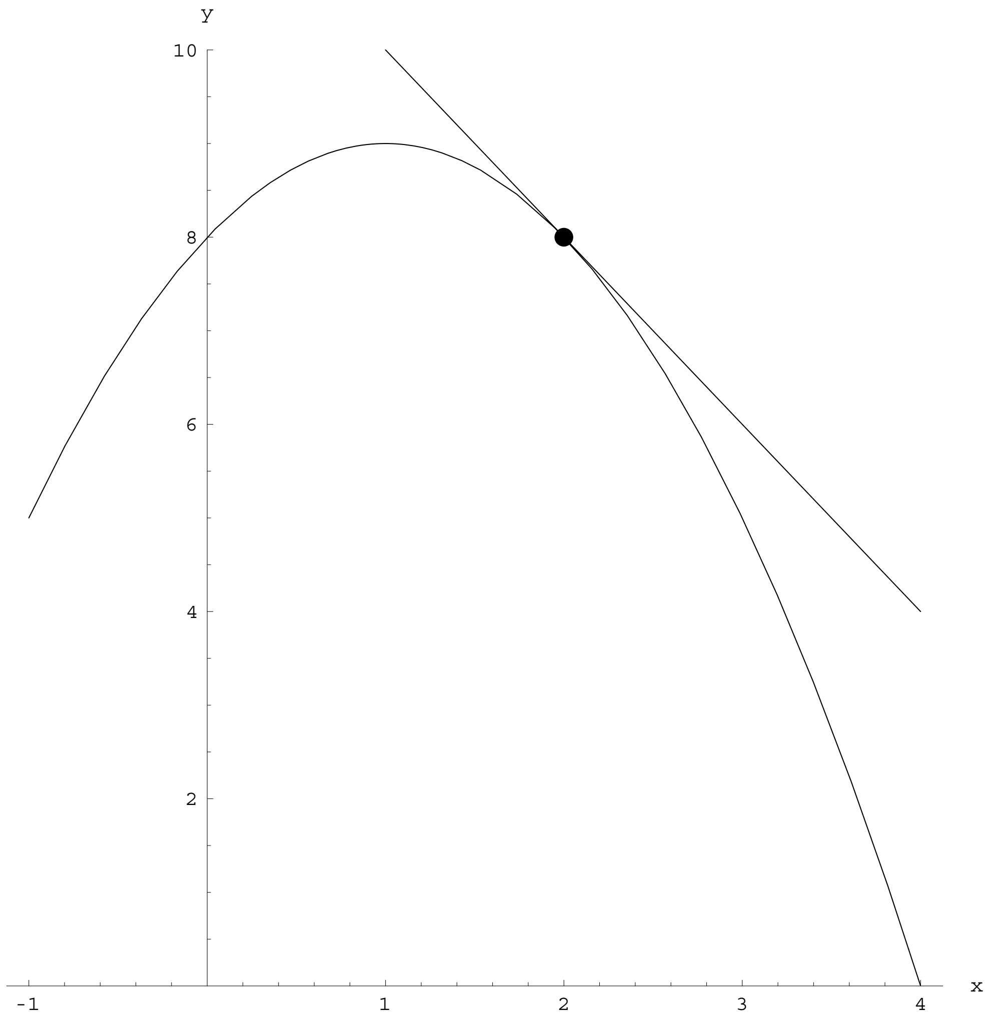
Graph of $z = x^3 - y^2$



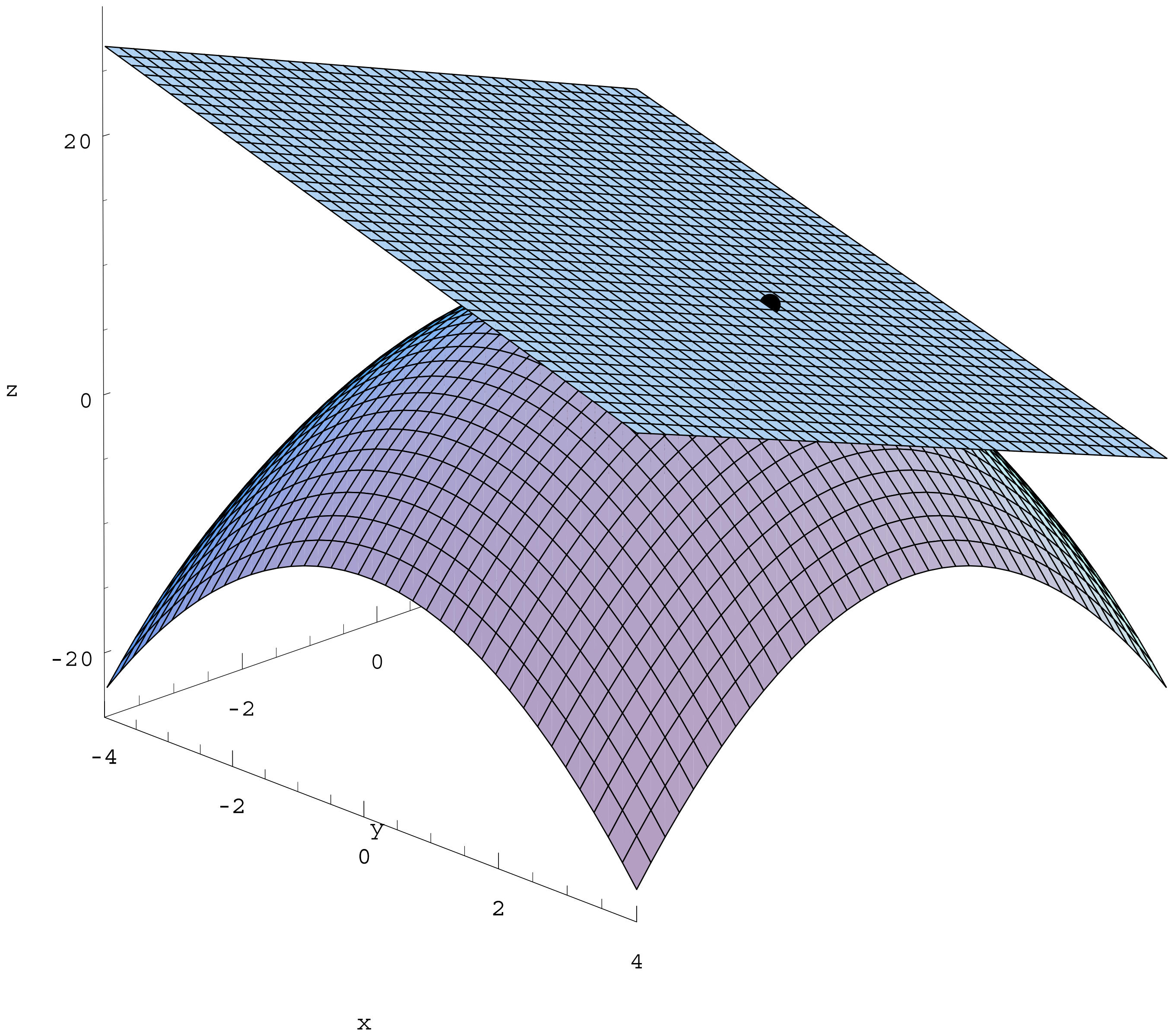
Contour Map of $z = x^3 - y^2$



In two dimensions we have the idea of tangent and the slope of the tangent (or derivative).



In three dimensions we have the idea of tangent plane and we need to find ideas which correspond to the slope of the tangent (or derivative). There are many different slopes at a point on a surface: imagine you are skiing; you can ski directly downhill or lessen the slope by skiing across the slope. All these possible slopes lie in a single plane, the tangent plane.


20
z
0
-20
0
-2
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-2
y
0
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x

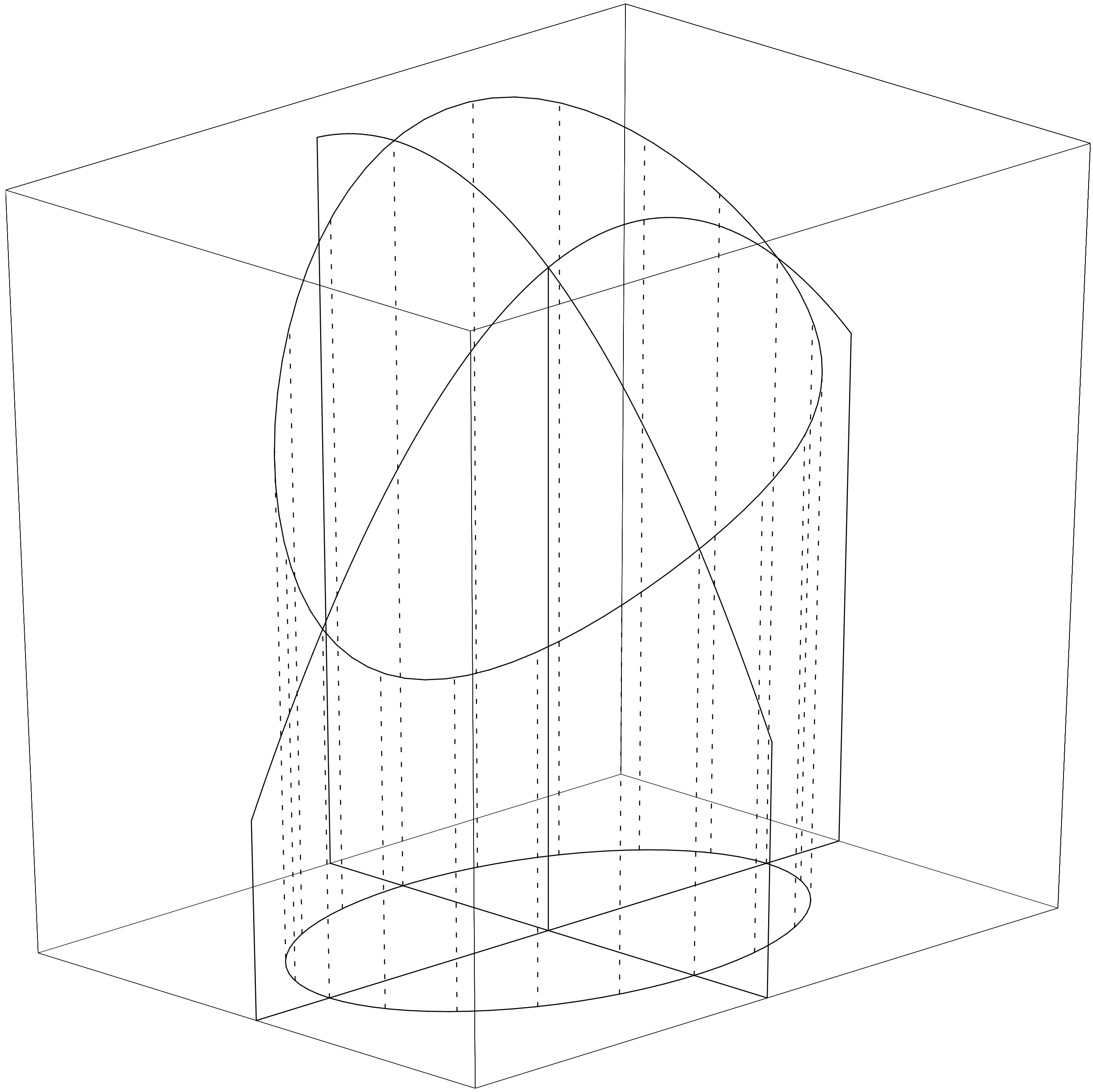
Clearly we want to be able to find tangent planes: they will provide the key to finding maxima and minima; the tangent plane going flat (as at the top of a hill or the bottom of a valley) is exactly like the tangent line going flat for functions of one variable.

Finding the tangent plane.

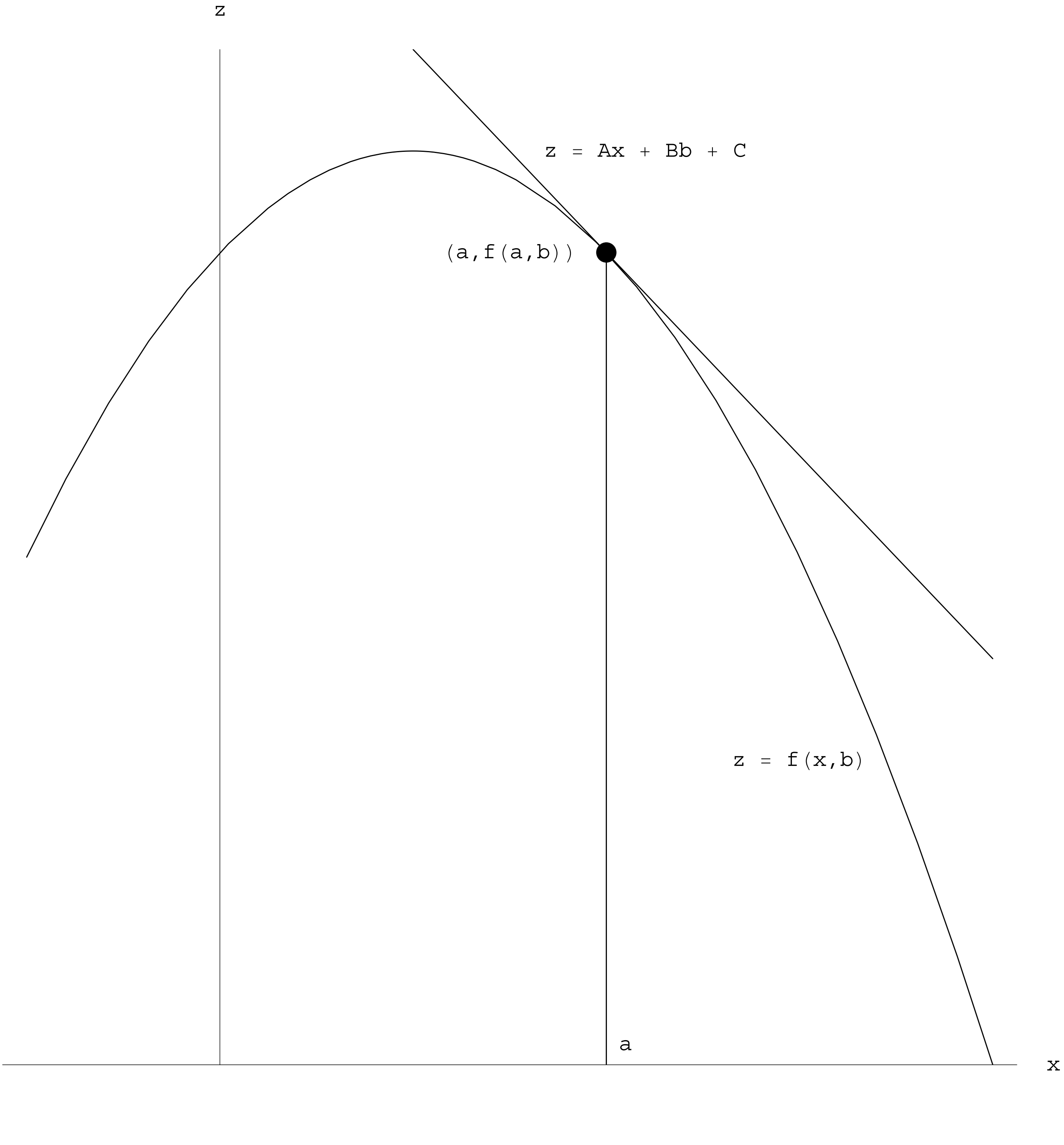
Take a function, $f(x, y)$, of two variables and take a point (a, b) in the xy-plane. Assume that the tangent plane has equation

$$z = Ax + By + C$$

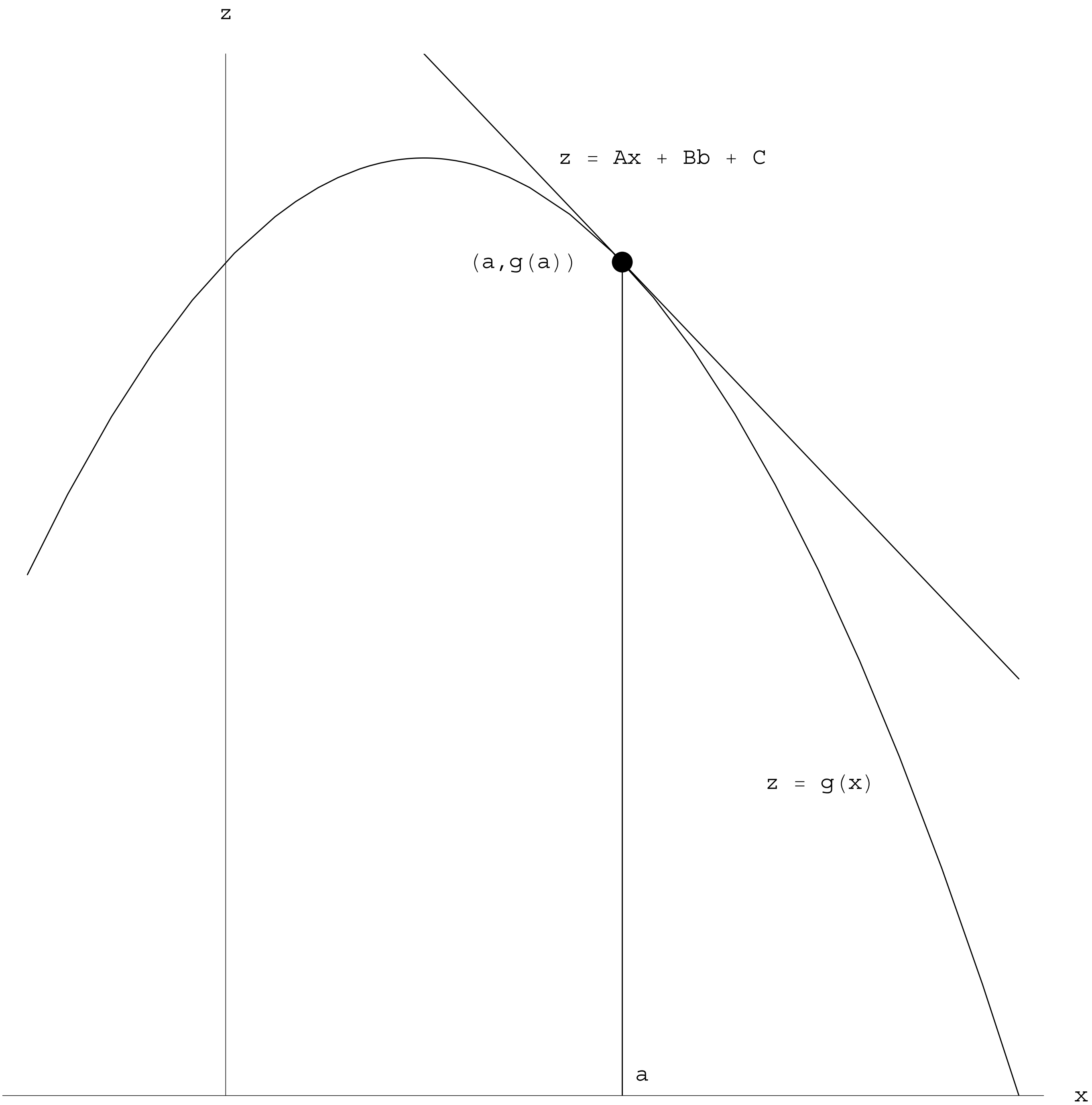
Look at the section where $y = b$: the section of the surface is the graph $z = f(x, b)$; the section of the tangent plane is the graph $z = Ax + Bb + C$.



We get the following familiar picture.



The section is a graph with a tangent line. We already know how to handle this. For clarity write $g(x) = f(x, b)$: so that:



We can now write down the equation of the tangent in two ways and compare them.
By sectioning

$$z = Ax + Bb + C.$$

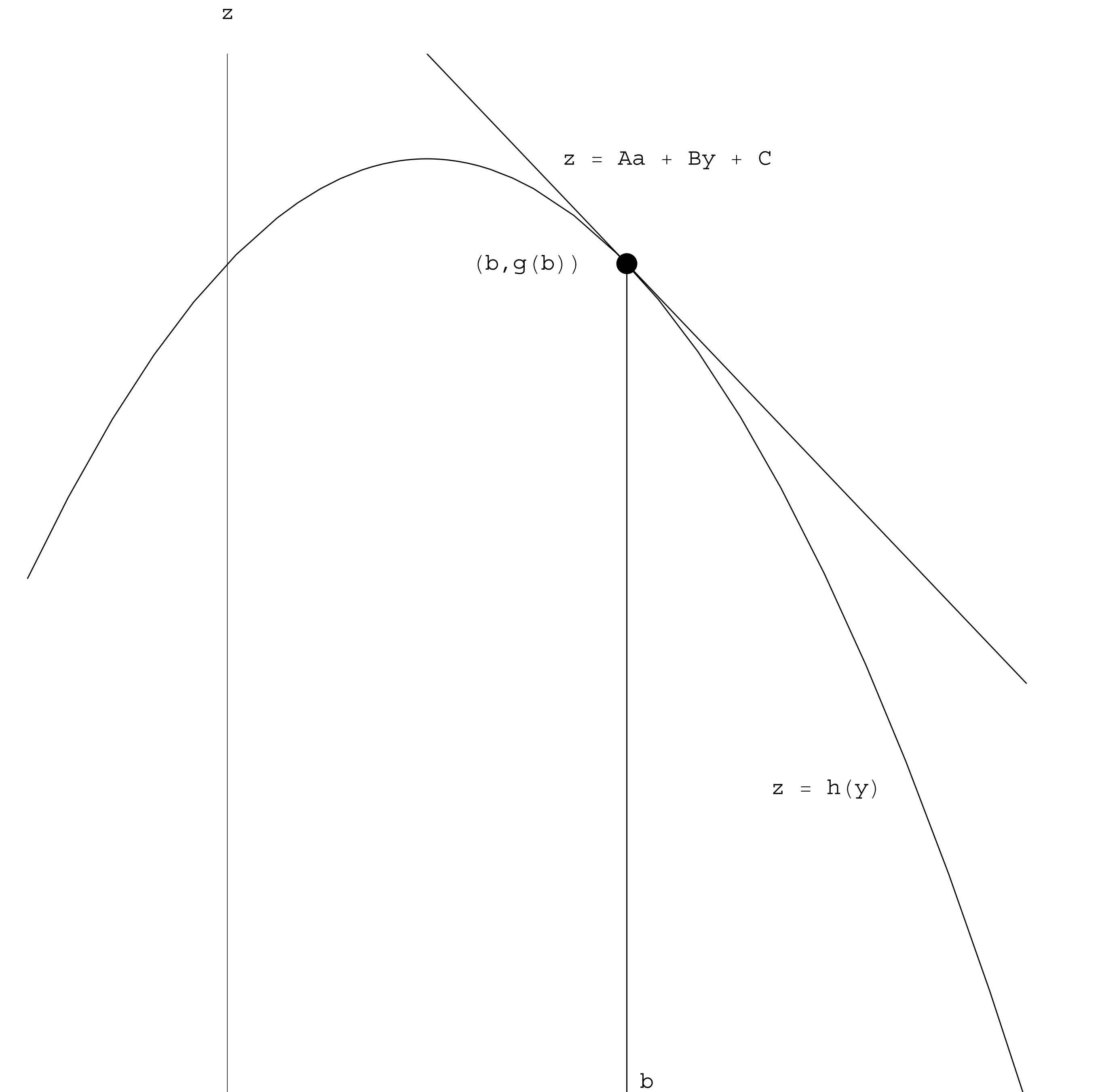
By using the last picture

$$\begin{aligned} z &= g(a) + g'(a)(x-a) \\ &= g'(a)x + g(a) - g'(a)a. \end{aligned}$$

So

$$\begin{aligned} A &= g'(a) \\ Bb + C &= g(a) - g'(a) \end{aligned}$$

Taking a similar section $x = a$ and writing $h(y) = f(a, y)$ we get



We can now write down the equation of the tangent in two ways and compare them.
By sectioning

$$z = Aa + By + C.$$

By using the last picture

$$\begin{aligned} z &= h(b) + h'(b)(y - b) \\ &= h'(b)x + h(b) - h'(b)b. \end{aligned}$$

So

$$\begin{aligned} B &= h'(b) \\ Aa + C &= h(b) - h'(b) \end{aligned}$$

Collecting we get

$$
\begin{aligned}
A &= g'(a) \\
B &= h'(b) \\
C &= h(b) - h'(b)b - Aa \\
&= h(b) - h'(b)b - g'(a)a \\
C &= g(a) - g'(a)a - Bb \\
&= g(a) - g'(a)a - h'(b)b
\end{aligned}
$$

The two versions of C are identical because

$$g(a) = f(a, b) = h(b).$$

The tangent plane is

$$z = g'(a)x + h'(b)y - g'(a)a - h'(b)b + f(a,b)$$

or

$$z - f(a,b) = g'(a)(x-a) + h'(b)(y-b).$$

What are g' and h'?

$g'(x)$ is obtained by holding y constant ($y = b$) and differentiating with respect to x.
$h'(y)$ is obtained by holding x constant ($x = a$) and differentiating with respect to y.

g' and h' are called *partial derivatives*

g' is written $\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$ or f_x.

h' is written $\dfrac{\partial z}{\partial y}$ or $\dfrac{\partial f}{\partial y}$ or f_y.

The tangent plane is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Example

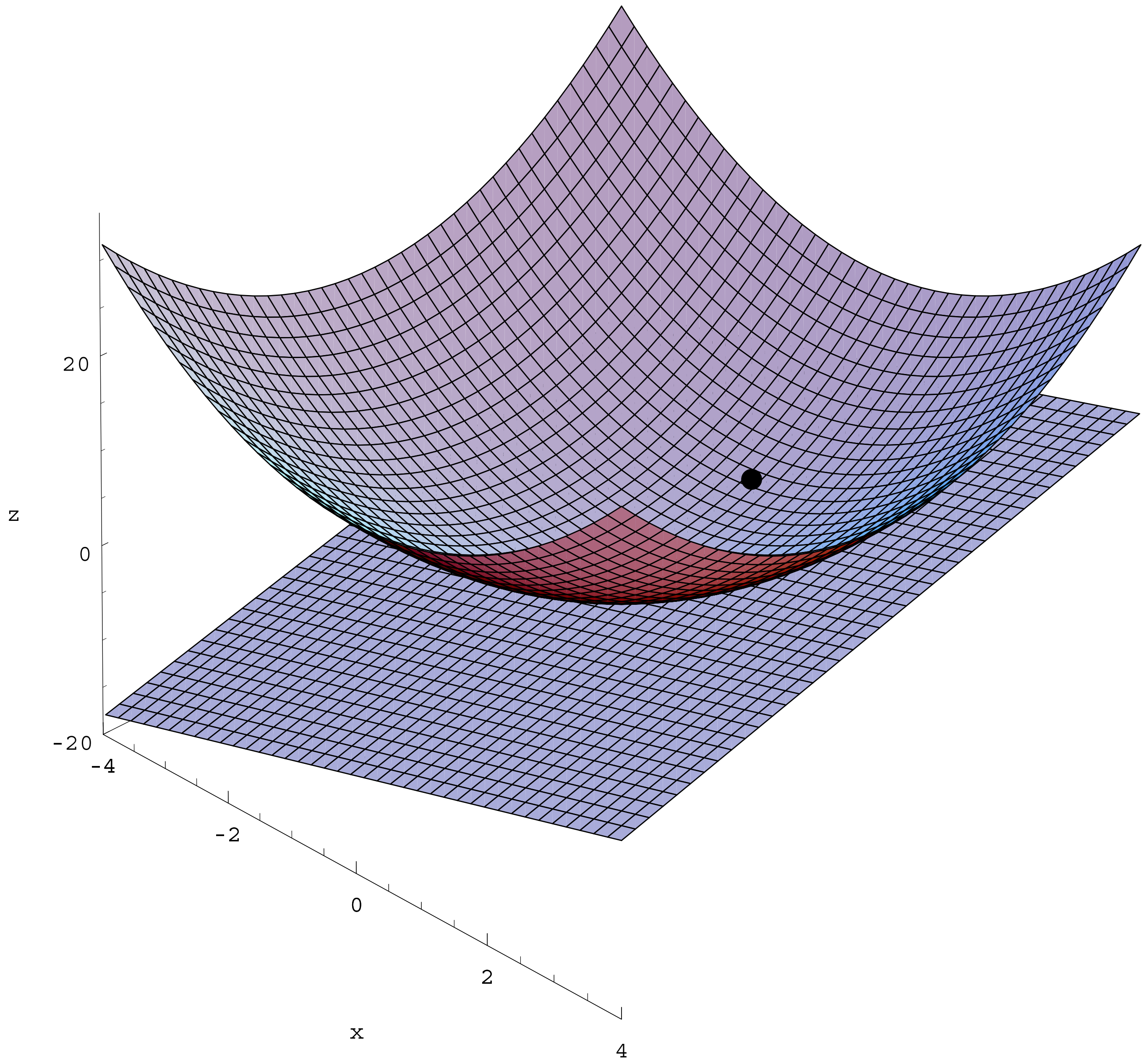
Find the tangent plane to $f(x,y) = x^2 + y^2$ at $(1,1,2)$.

$f_x(x,y) = 2x$ and $f_y(x,y) = 2y$. So $f_x(1,1) = 2$ and $f_y(1,1) = 2$ and the equation of the tangent plane is

$$z - 2 = 2(x-1) + 2(y-1)$$

or

$$2x + 2y - z = 2$$


20
z
0
-20
-4
-2
0
2
x
4

So if the problem is to *find the maximum or minimum values of* $z = f(x, y)$ *where* (x, y) *runs over a region* R *of the plane*, we can follow the pattern of the one variable situation. You can forget about the points where the tangent is not horizontal. So the points where absolute maxima and absolute minima might occur are:

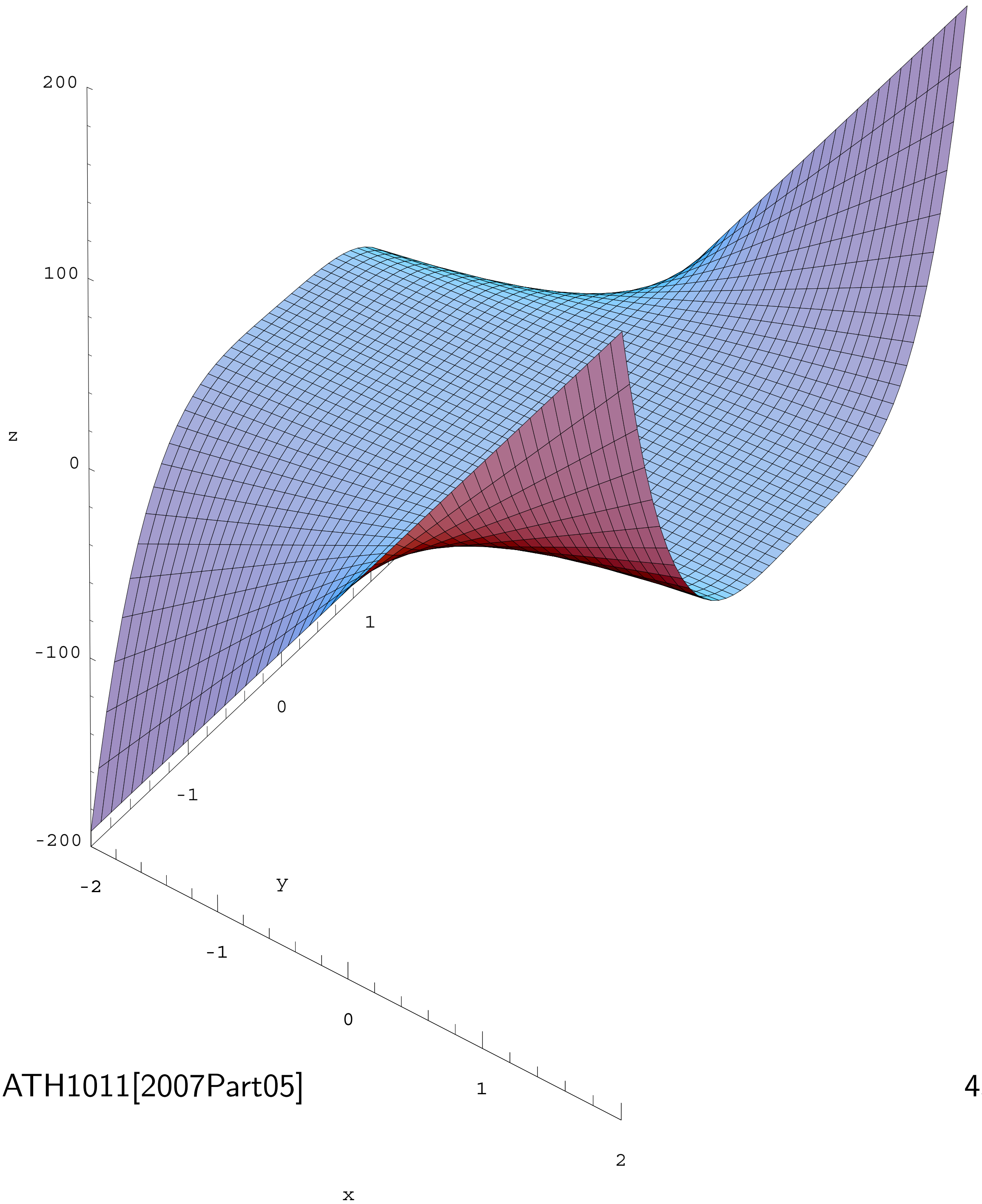
- points where the tangent plane is horizontal ($f_x = f_y = 0$),
- points where the tangent plane is undefined,
- points on the boundary of the region R.

Having to think about the boundary of R makes the two variable problem much harder than the one variable problem.

We need to practise finding partial derivatives.

Example

Let $f(x, y) = 6xy^4$.



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$f_x(x, y) = 6y^4$.

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$f_y(x, y) = 6x \times 4y^3 = 24xy^3$.

$f_x(1, 2) = 96 \qquad f_y(1, 2) = 192$.

Tangent plane at $(1, 2, 96)$ is

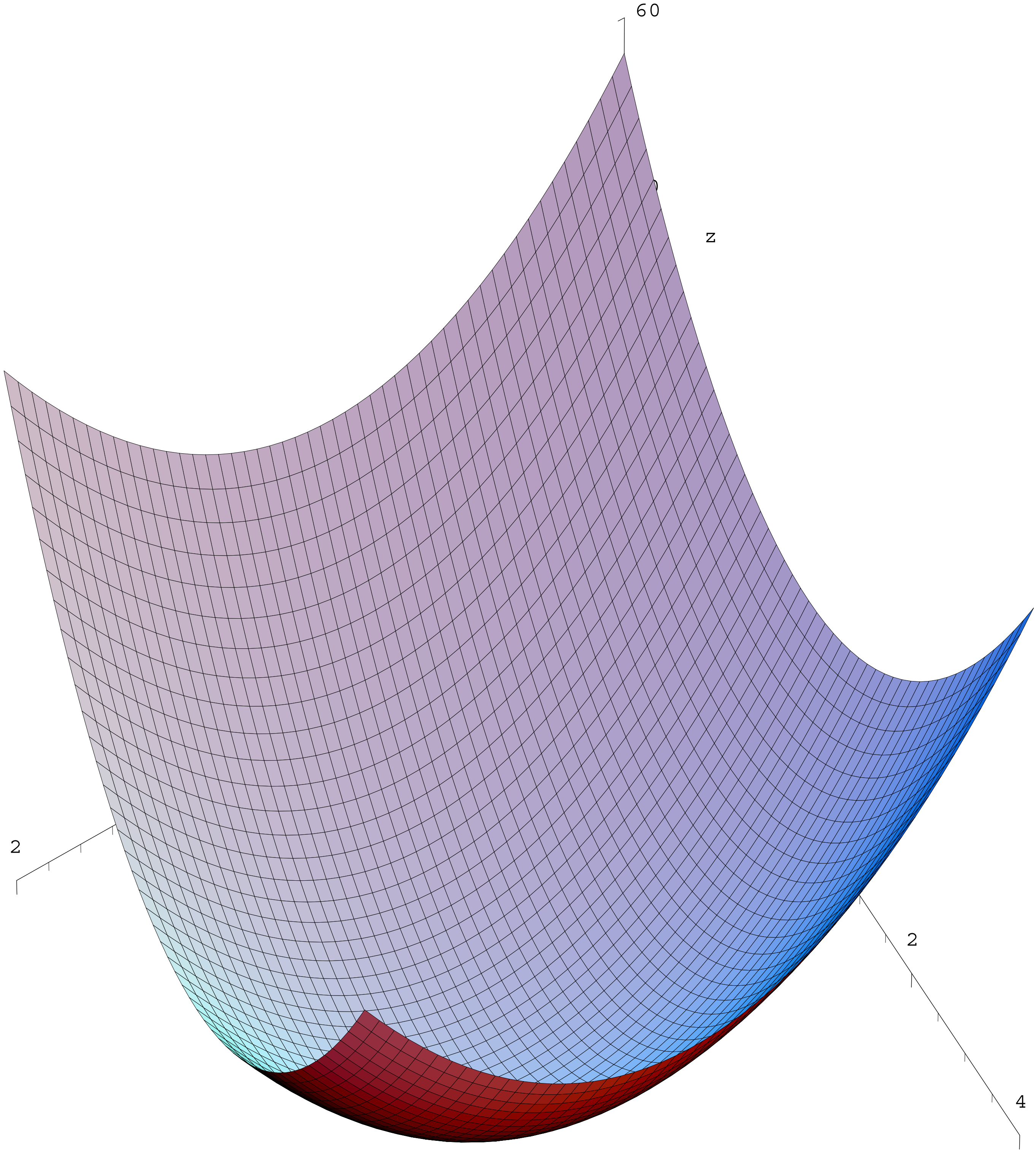
$$z - 96 = 96(x - 1) + 192(y - 2)$$

i.e.

$$96x + 192y - z = 384.$$

Example

Let $f(x, y) = 6x^2 + 2y^2$.



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$f_x(x, y) = 6 \times 2x = 12x$.

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$f_y(x, y) = 2 \times 2y = 4y$.

$f_x(1, -1) = 12 \qquad f_y(1, -1) = -4$.

Tangent plane at $(1, -1, 8)$ is

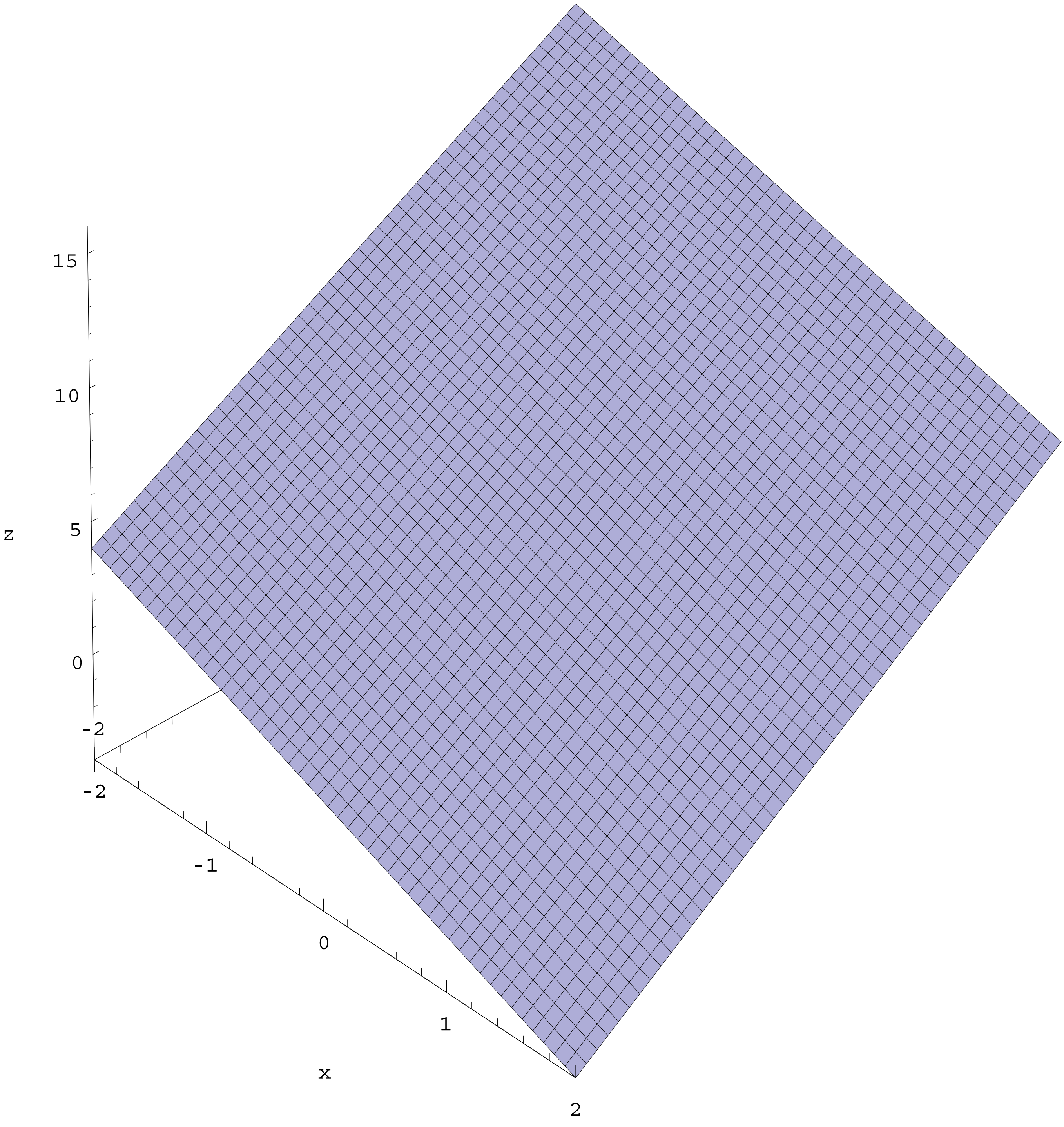
$$z - 8 = 12(x - 1) - 4(y + 1)$$

i.e.

$$12x - 4y - z = 8$$

Example

Let $f(x, y) = -2x + 3y + 6$.



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$f_x(x, y) = -2$.

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$f_y(x, y) = 3$.

$f_x(1, 1) = -2 \qquad f_y(1, 1) = 3$.

Tangent plane at $(1, 1, 7)$ is

$$z - 7 = -2(x - 1) + 3(y - 1)$$

i.e.

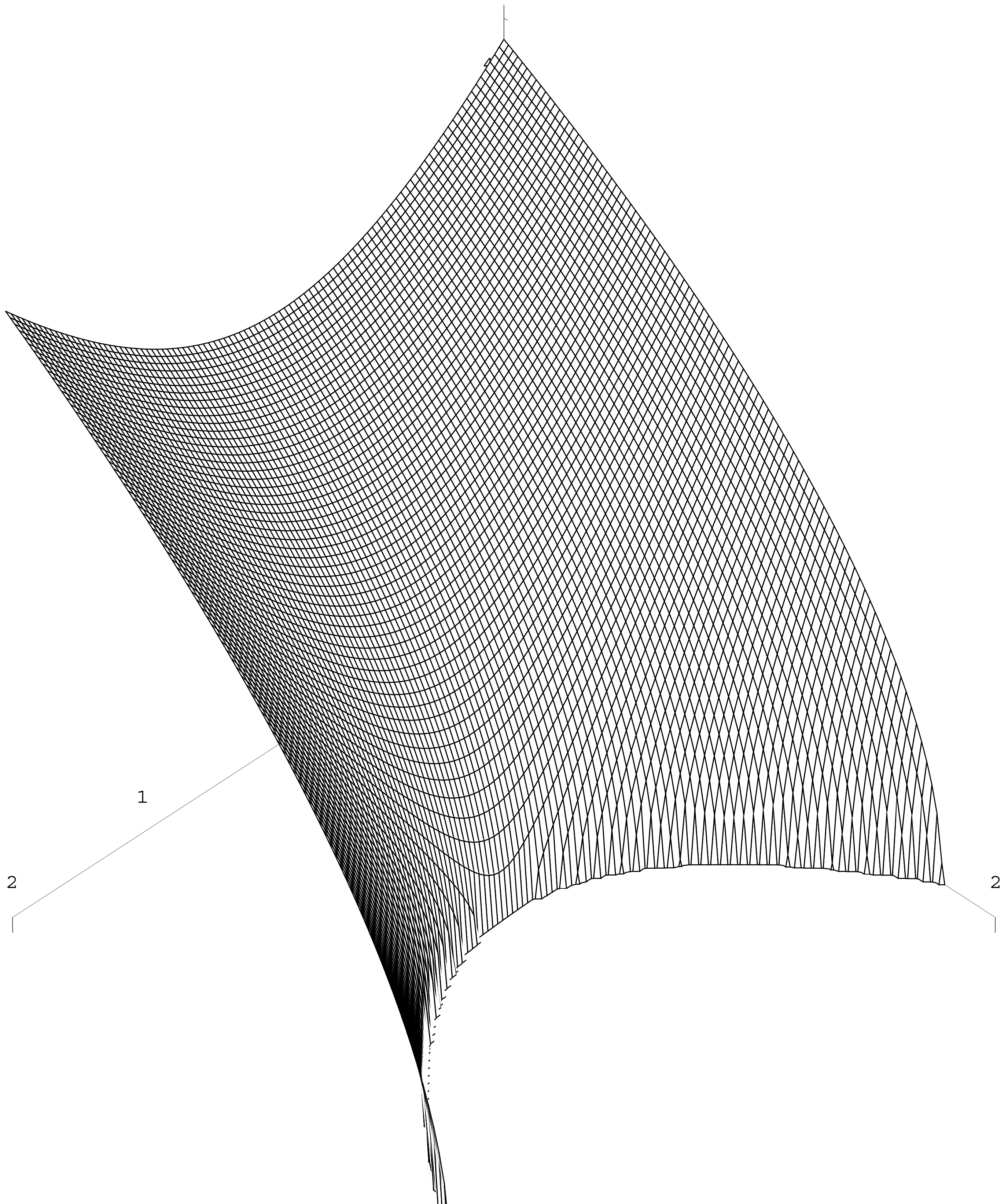
$$-2x + 3y - z = 6$$

A plane is its own tangent at all points in the plane.

Example

Let

$$\begin{aligned} f(x,y) &= \sqrt{2x^2 - 5y} \\ &= \left(2x^2 - 5y\right)^{\frac{1}{2}}. \end{aligned}$$


1
2
2

To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$$f_x(x,y) = \tfrac{1}{2}\left(2x^2 - 5y\right)^{-\frac{1}{2}} \times 4x$$
$$= \frac{2x}{\sqrt{2x^2 - 5y}}.$$

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$$f_y(x,y) = \tfrac{1}{2}\left(2x^2 - 5y\right)^{-\frac{1}{2}} \times (-5)$$
$$= -\frac{5}{\sqrt{2x^2 - 5y}}.$$

$$f_x(2,0) = \sqrt{2} \qquad f_y(2,0) = -\tfrac{5}{4}\sqrt{2}.$$

Tangent plane at $(2, 0, \sqrt{2})$ is

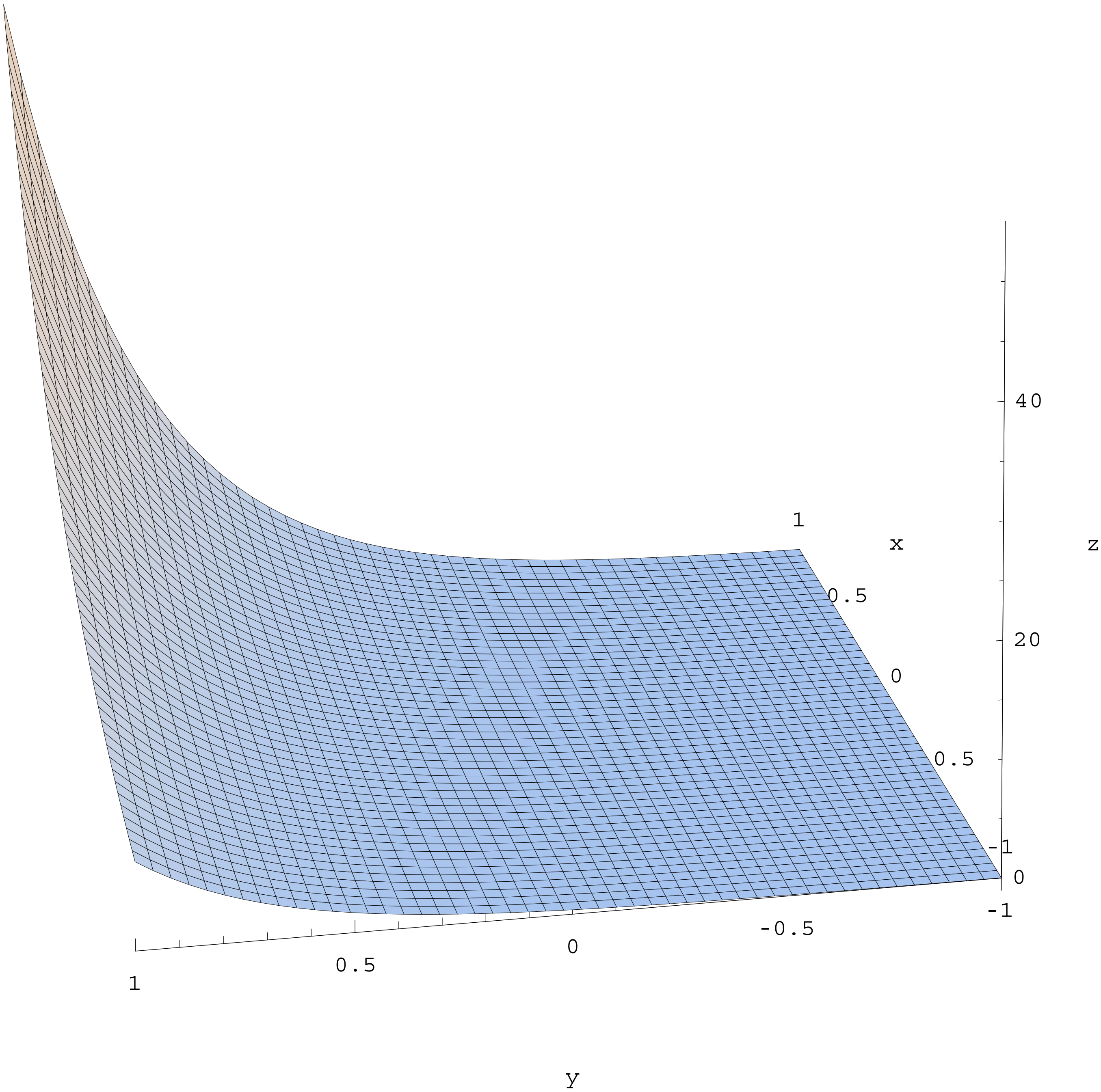
$$z - \sqrt{2} = \sqrt{2}(x-2) - \tfrac{5}{4}\sqrt{2}(y-0)$$

i.e.

$$4x - 5y - 2\sqrt{2}z = 4$$

Example

Let $f(x, y) = e^{x+3y}$.



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$f_x(x, y) = e^{x+3y} \times 1 = e^{x+3y}$.

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$f_y(x, y) = e^{x+3y} \times 3 = 3e^{x+3y}$.

$f_x(1, 1) = e^4 \qquad f_y(1, 1) = 3e^4$.

Tangent plane at $(1, 1, e^4)$ is

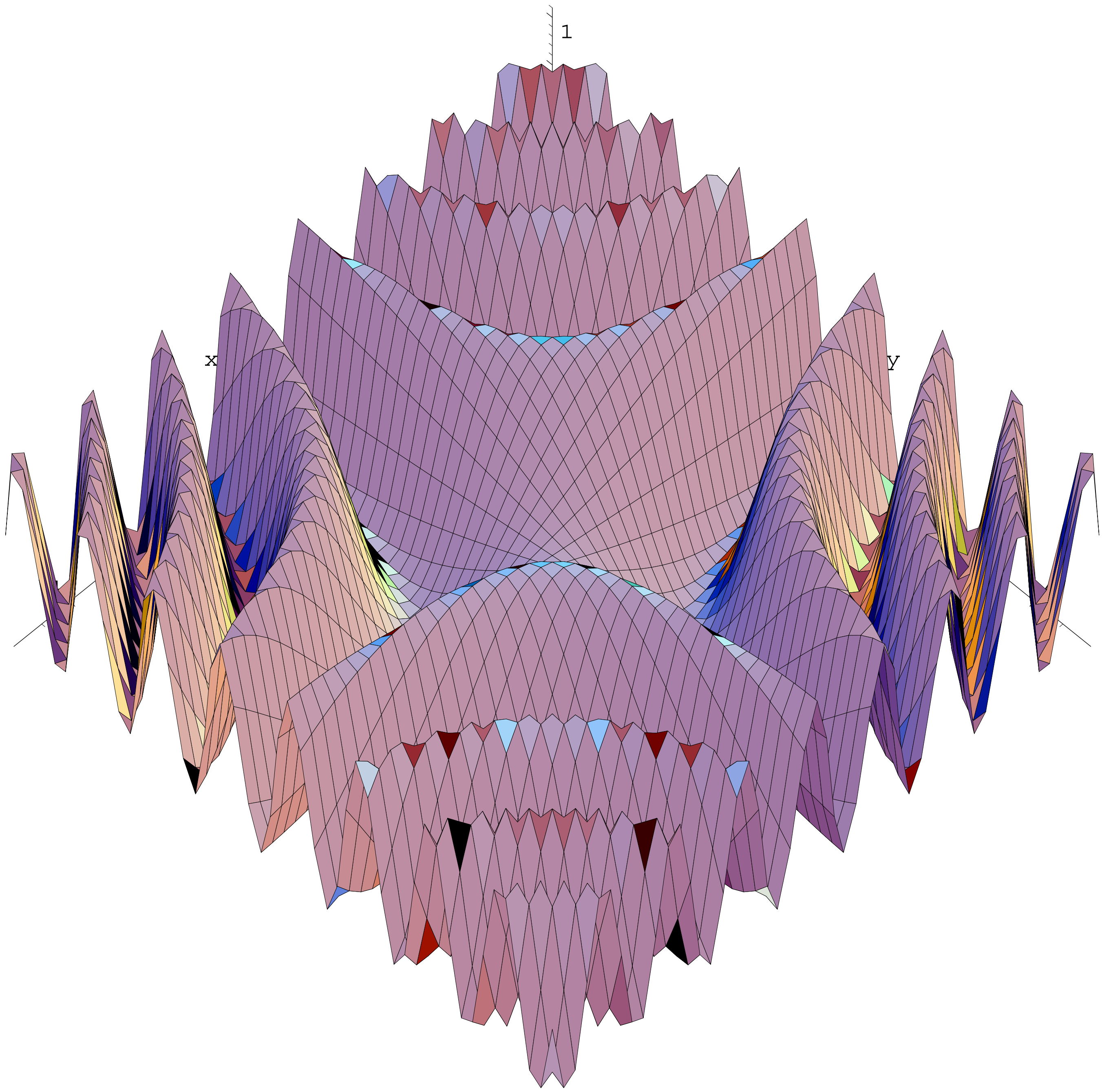
$$z - e^4 = e^4(x - 1) + 3e^4(y - 1)$$

i.e.

$$e^4x + 3e^4y - z = 3e^4$$

Example

Let $f(x, y) = \sin(xy)$.



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$f_x(x, y) = \cos(xy) \times y = y \cos(xy)$.

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$f_y(x, y) = \cos(xy) \times x = x \cos(xy)$.

$f_x(1, \frac{\pi}{2}) = 0 \qquad f_y(1, \frac{\pi}{2}) = 0$.

Tangent plane at $(1, \frac{\pi}{2}, 1)$ is

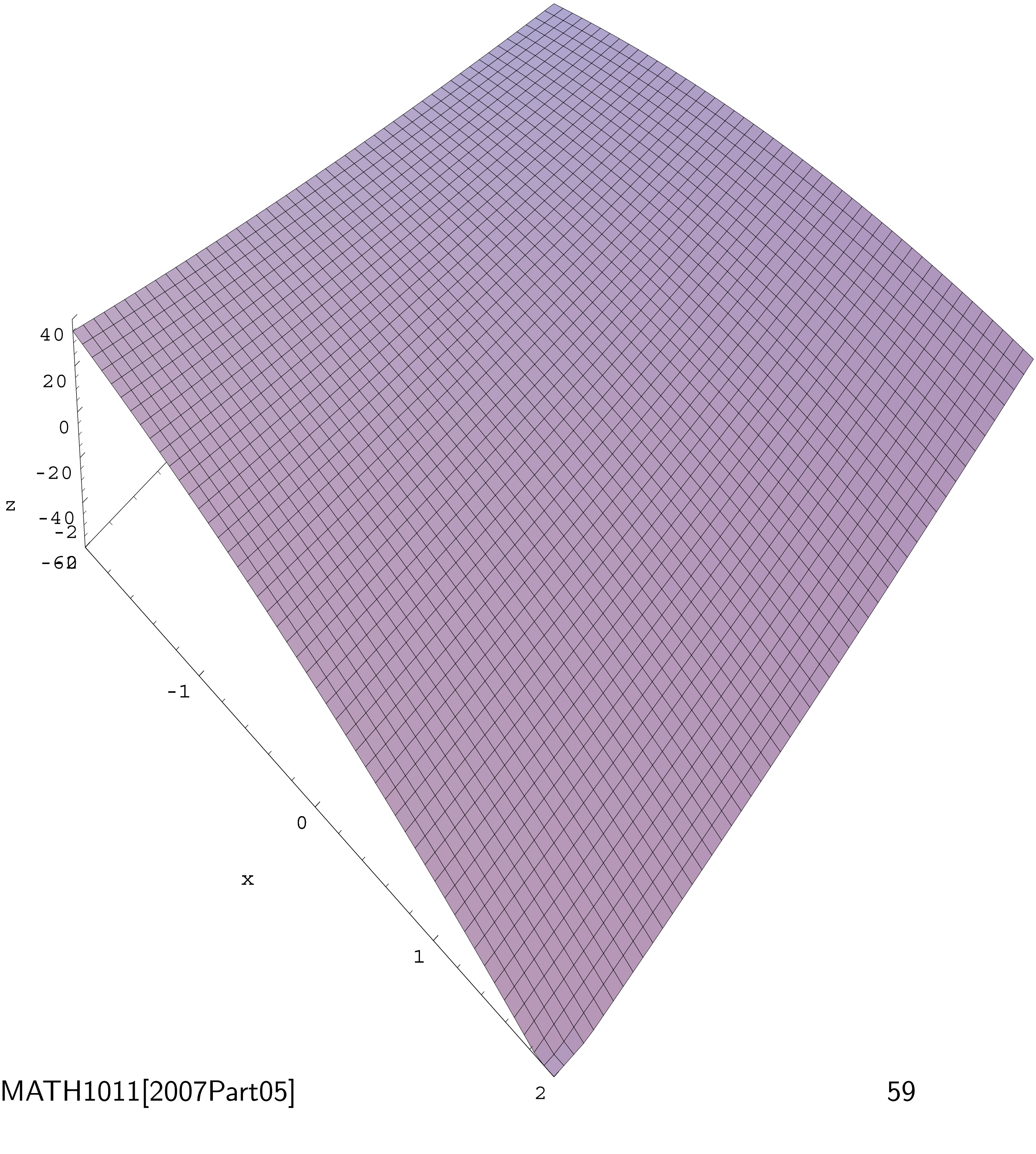
$$z - 1 = 0(x - 1) + 0(y - \tfrac{\pi}{2})$$

i.e.

$$z = 1$$

Example

Let $f(x, y) = 10xy - 3x^2 + y^2 - 5x + 2y - 3$.



To find f_x: think *“differentiating with respect to x, so hold all other variables constant.”*

$f_x(x, y) = 10y - 6x - 5.$

To find f_y: think *“differentiating with respect to y, so hold all other variables constant.”*

$f_y(x, y) = 10x + 2y + 2.$

To find the places where the tangent goes horizontal, we set

$$f_x(x, y) = f_y(x, y) = 0.$$

Then

$$\begin{array}{rcrcrcl} -\;\; 6x & + & 10y & - & 5 & = & 0 \\ 10x & + & 2y & + & 2 & = & 0 \end{array}$$

i.e

$$x = \frac{15}{56}$$

$$y = \frac{19}{56}$$

$f_x\left(\frac{15}{56}, \frac{19}{56}\right) = 0 \qquad f_y\left(\frac{15}{56}, \frac{19}{56}\right) = 0.$

Tangent plane at $\left(\frac{15}{56}, \frac{19}{56}, \frac{3483}{1568}\right)$ is

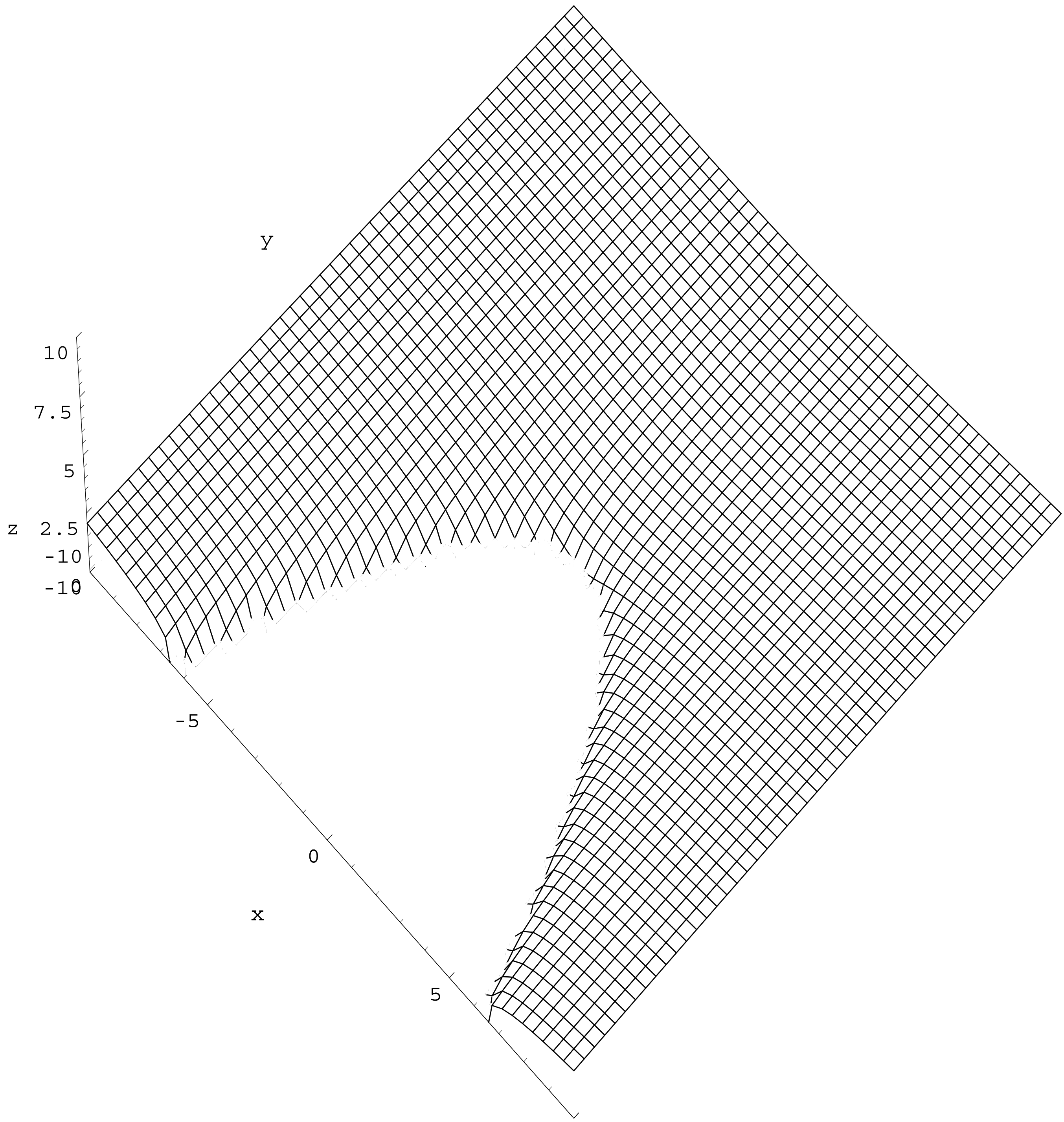
$$z - \frac{3483}{1568} = 0$$

i.e.

$$z = \frac{3483}{1568}$$

Example

$$f(x,y) = \ln\left(\sqrt{x^2+4y}\right)$$
$$= \tfrac{1}{2}\ln(x^2+4y)$$



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$$f_x(x,y) = \tfrac{1}{2} \times \frac{1}{x^2+4y} \times 2x$$
$$= \frac{x}{x^2+4y}$$

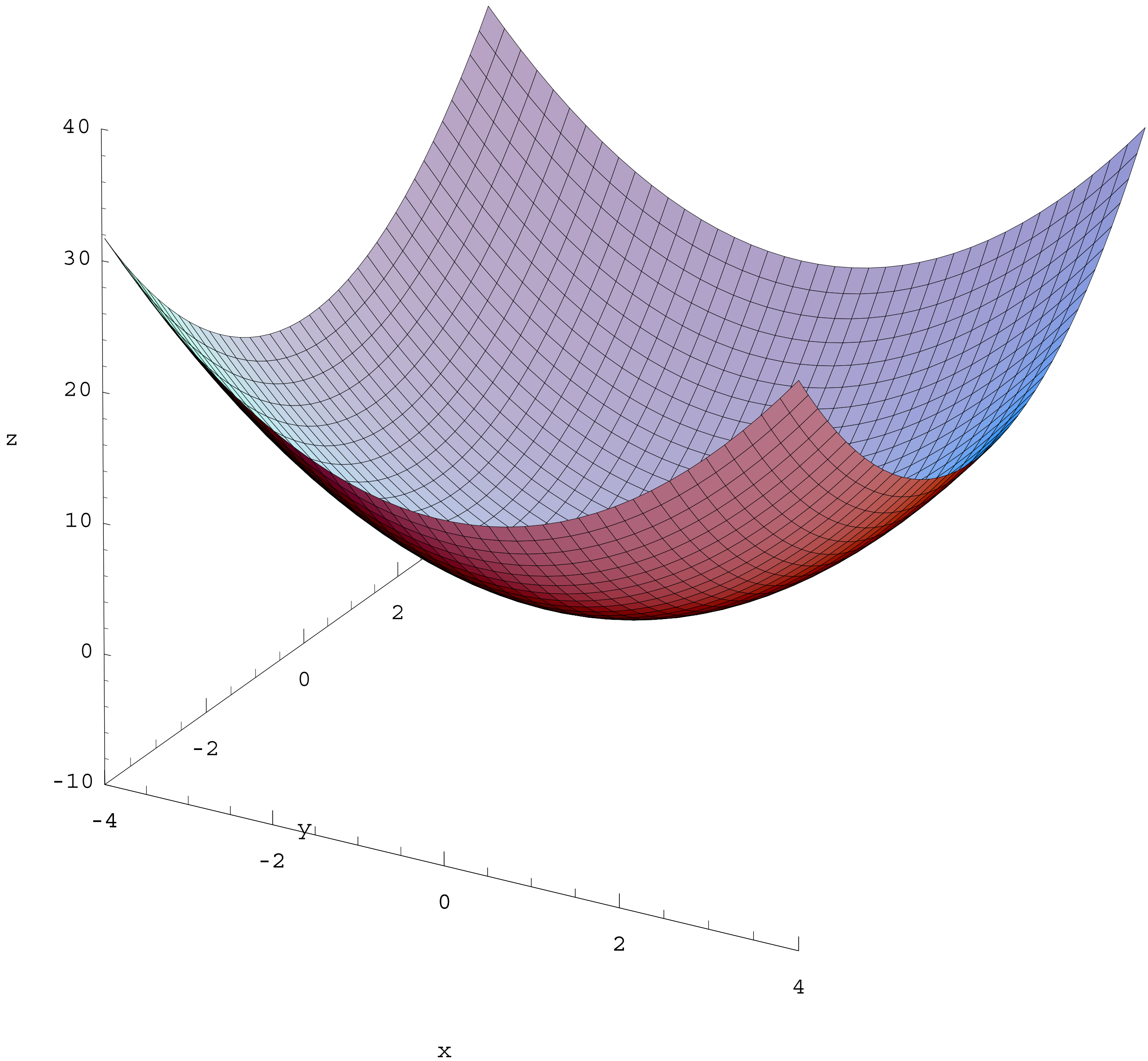
To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$$f_y(x,y) = \tfrac{1}{2} \times \frac{1}{x^2+4y} \times 4$$
$$= \frac{2}{x^2+4y}$$

There are no points where the tangent goes horizontal.

Example

Let $f(x, y) = x^2 + y^2$.



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*
$f_x(x, y) = 2x$

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*
$f_y(x, y) = 2y$

To find the places where the tangent goes horizontal, we set

$$f_x(x, y) = f_y(x, y) = 0.$$

Then

$$\begin{aligned} 2x &= 0 \\ 2y &= 0 \end{aligned}$$

i.e. $x = y = 0$. So the tangent plane

is horizontal at the origin and only at the origin.

The tangent plane at (x_0, y_0, z_0) $[z_0 = x_0^2 + y_0^2]$ is

$$z - z_0 = 2x_0(x - x_0) + 2y_0(y - y_0)$$

i.e.

$$z - x_0^2 - y_0^2 = 2x_0x - 2x_0^2 + 2y_0y - 2y_0^2$$

i.e.

$$z = 2x_0x + 2y_0y - y_0^2 - x_0^2$$

i.e.

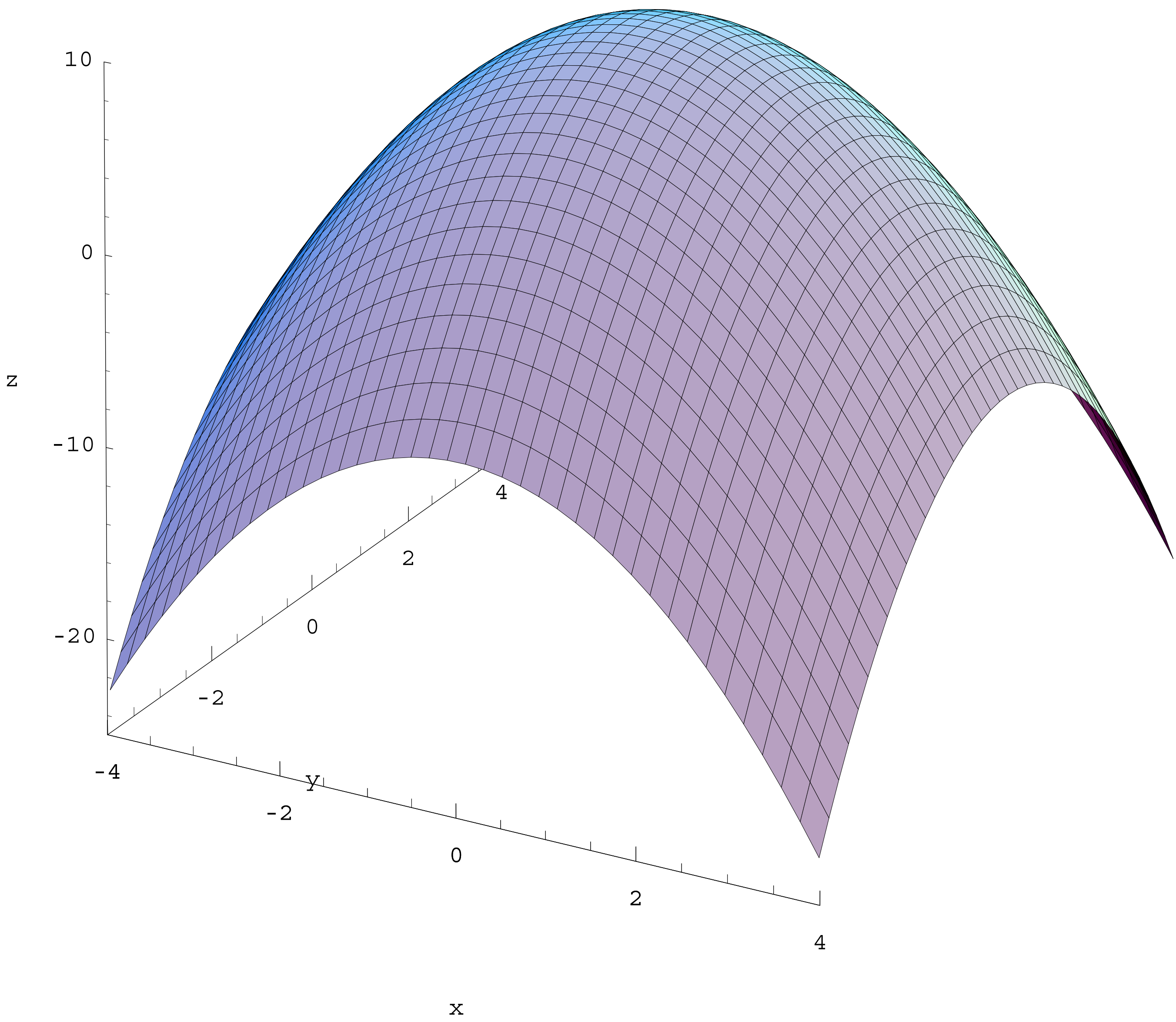
$$z + z_0 = 2x_0x + 2y_0y$$

Example

Let $f(x, y) = 9 - x^2 - y^2$ and

$$R = \{(x, y) : x^2 + y^2 \leq 9\}$$

(i.e. R is the interior and circumference of the circle centred at the origin of radius 3. If we are required to maximise f over the region R, we note that $f(x, y) = 0$ on the boundary of R and positive on the interior. So the maximum is in the interior of R and as we shall see below can only occur at a point where $f_x(x, y) = f_y(x, y) = 0$.


10
0
z
-10
-20
4
2
0
-2
y
-4
-2
0
2
4
x

To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*
$f_x(x, y) = -2x$

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*
$f_y(x, y) = -2y$

To find the places where the tangent goes horizontal, we set

$$f_x(x, y) = f_y(x, y) = 0.$$

Then

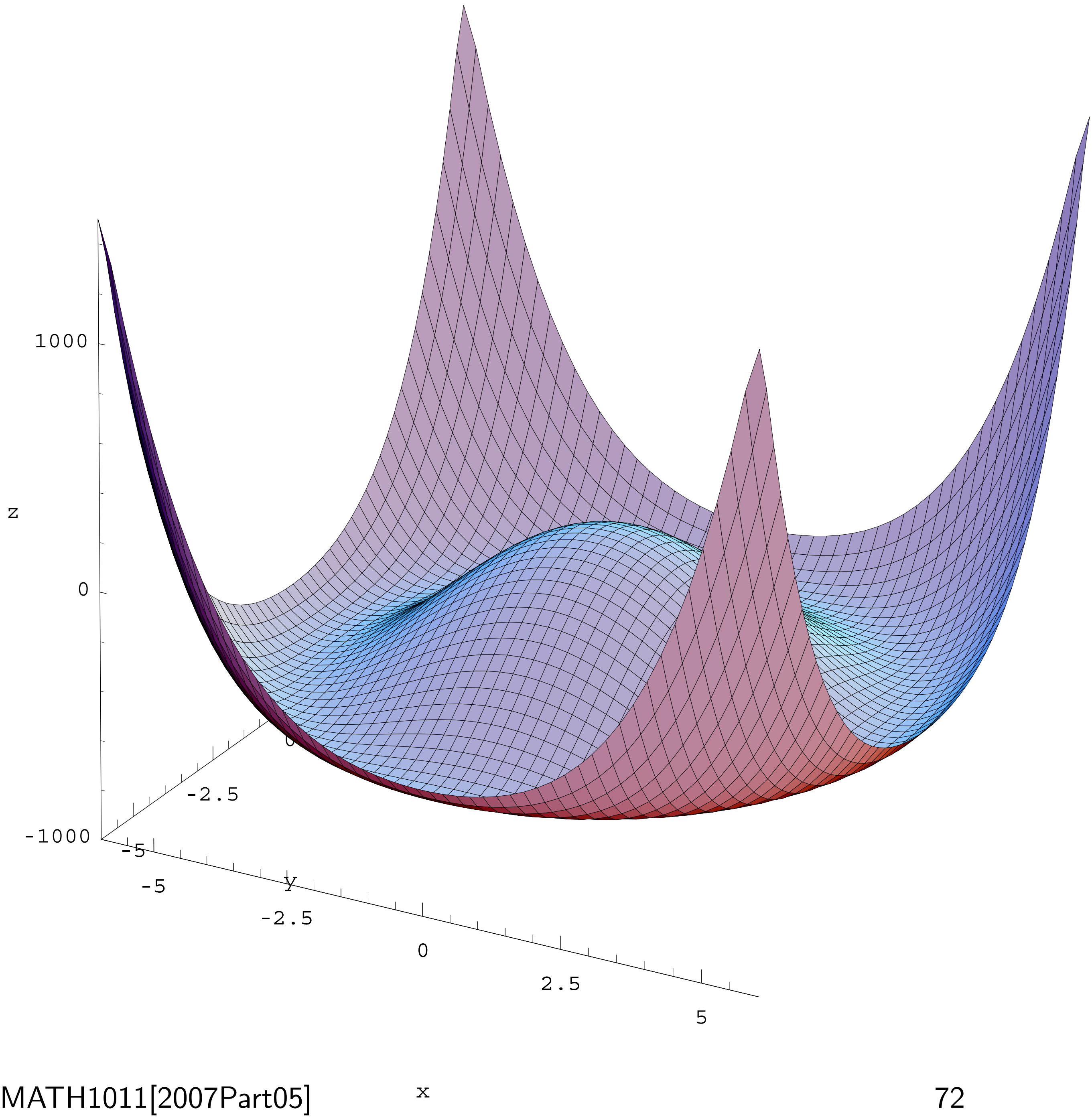
$$\begin{aligned} 2x &= 0 \\ 2y &= 0 \end{aligned}$$

i.e. $x = y = 0$. So the tangent plane is horizontal at the origin and only at the origin. There are no points where the tangent plane is undefined and there is no maximum on the boundary of R. Thus the maximum must ocur at the origin and the maximum value is 9.

Example

Let

$$f(x,y) = \left(x^2 + y^2\right)^2 - 50(x^2 + y^2) + 4.$$



To find f_x: think *"differentiating with respect to x, so hold all other variables constant."*

$$\begin{aligned} f_x(x,y) &= 2\left(x^2+y^2\right) \times 2x - 50(2x) \\ &= 4x(x^2+y^2-25) \end{aligned}$$

To find f_y: think *"differentiating with respect to y, so hold all other variables constant."*

$$\begin{aligned} f_y(x,y) &= 2\left(x^2+y^2\right) \times 2y - 50(2y) \\ &= 4y(x^2+y^2-25) \end{aligned}$$

To find the places where the tangent goes horizontal, we set

$$f_x(x,y) = f_y(x,y) = 0.$$

Then

$$4x(x^2 + y^2 - 25) = 0$$
$$4y(x^2 + y^2 - 25) = 0.$$

i.e. $x = y = 0$ or $(x^2 + y^2 - 25)$. So the tangent plane is horizontal at the origin and at each point on the circle of radius 5 centred at the origin.

When dealing with one variable optimisation, we paused to make a stab at classifying points where $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = 0$.

$\dfrac{dy}{dx} > 0$	f increases	
$\dfrac{dy}{dx} < 0$	f decreases	
$\dfrac{dy}{dx} = 0$	$\dfrac{d^2y}{dx^2} < 0$	local maximum
	$\dfrac{d^2y}{dx^2} > 0$	local minimum
	$\dfrac{d^2y}{dx^2} = 0$	more investigation

$\dfrac{d^2y}{dx^2} > 0$	f is concave up
$\dfrac{d^2y}{dx^2} < 0$	f is concave down
$\dfrac{d^2y}{dx^2} = 0$	f may have an inflexion

We now look at the analogous problem for functions of two variables. First we need to look at second order partial derivatives.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = (z_x)_x = z_{xx}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = (z_y)_y = z_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = (z_y)_x = z_{yx}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = (z_x)_y = z_{xy}$$

For all the functions that we look at it will be true that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

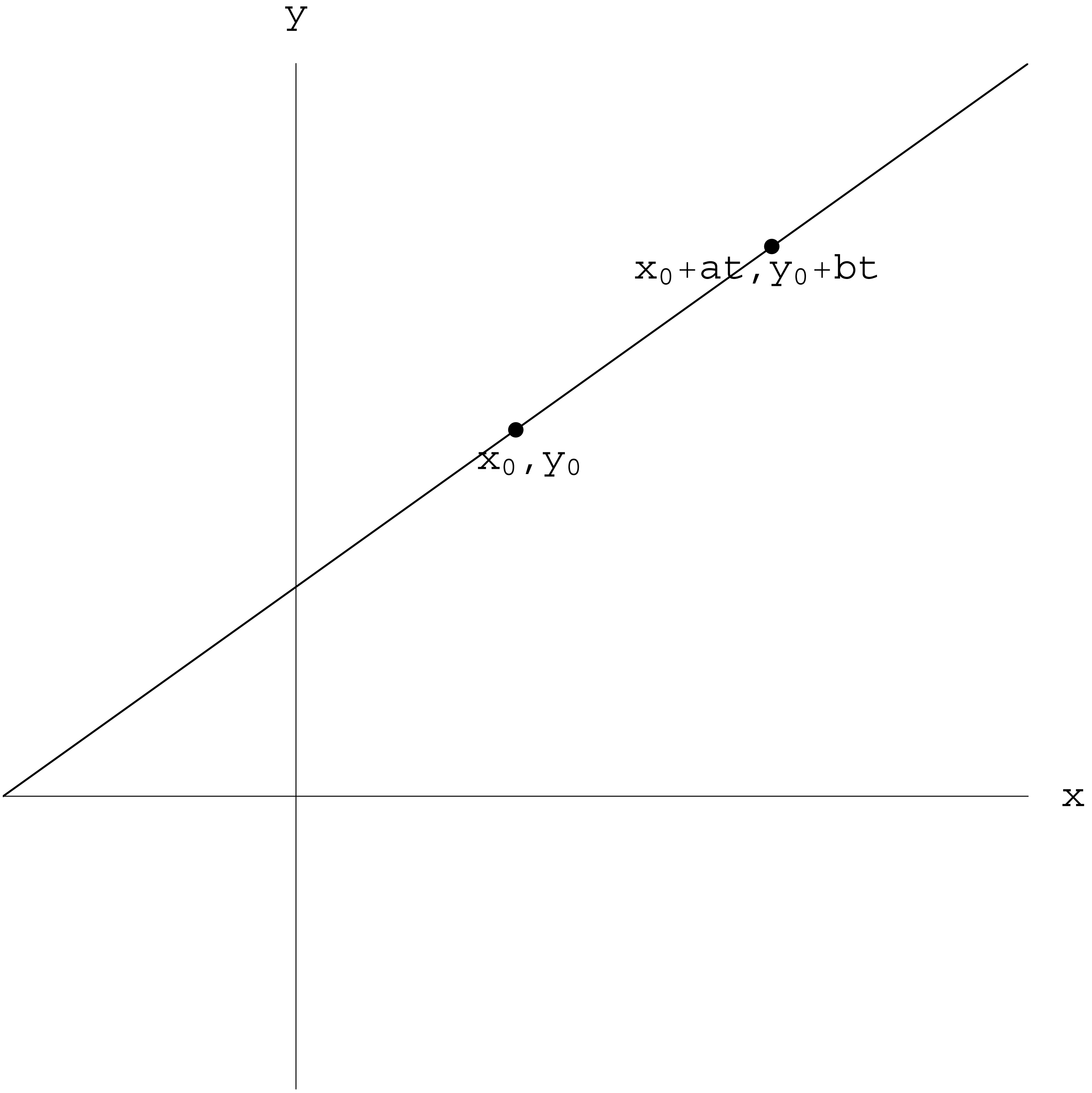
We shall check once and then take it for granted from then on.

Example
Find the first and second order partial derivatives of $z = 5x^3y^2 - 7xy$.

$$\begin{aligned} z_x &= 15x^2y^2 - 7y & z_y &= 10x^3y - 7x \\ z_{xx} &= 30xy^2 & z_{yy} &= 10x^3 \\ z_{xy} &= 30x^2y - 7 & z_{yx} &= 30x^2y - 7 \end{aligned}$$

We now look for conditions for $z = f(x, y)$ to have local maxima, local minima and saddles.

Suppose we have $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Look at vertical sections through (x_0, y_0).


y
x
x0+at,y0+bt
x0,y0

In this section, the curve is

$$z = f(x_0 + at, y_0 + bt).$$

which means that

$$\begin{aligned}\frac{dz}{dt} &= f_x(x_0 + at, y_0 + bt) \times a \\ &\qquad + f_y(x_0 + at, y_0 + bt) \times b \\ &= af_x + bf_y\end{aligned}$$

and

$$\frac{d^2z}{dt^2} = a^2 f_{xx} + 2ab f_{xy} + b^2 f_{yy}$$

To get a local maximum we need a local maximum in each section (for all choices of a and b). So, for all choices of a and b, we need $\frac{dz}{dt} = 0$ and $\frac{d^2z}{dt^2} \leq 0$.

To get a local minimum we need a local minimum in each section (for all choices of a and b). So, for all choices of a and b, we need $\frac{dz}{dt} = 0$ and $\frac{d^2z}{dt^2} \geq 0$.

The $\frac{dz}{dt} = 0$ for all choices of a and b means that we need $f_x = f_y = 0$.

Completing the square we see that

$$\frac{d^2z}{dt^2} = a^2 f_{xx} + 2ab f_{xy} + b^2 f_{yy}$$
$$= f_{xx}\left(a + \frac{b f_{xy}}{f_{xx}}\right)^2$$
$$+ \frac{b^2}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}^2\right)$$

This means that we can ensure a local maximum with the conditions:

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0$$

and a local minimum with the conditions:

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0$$

Similar calculations show that alternative conditions are, for a local maximum

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{yy} < 0$$

and for a local minimum

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{yy} > 0$$

Further investigation shows that

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 < 0$$

gives a saddle.

When

$$\Delta = f_{xx}f_{yy} - f_{xy}^2 = 0$$

all sort of interesting things might happen.

One variable local maxima and minima

$\dfrac{dy}{dx} = 0$	$\dfrac{d^2y}{dx^2} < 0$	local maximum
	$\dfrac{d^2y}{dx^2} > 0$	local minimum
	$\dfrac{d^2y}{dx^2} = 0$	more investigation

Two variable local maxima and minima

Suppose $f_x = f_y = 0$ and

$$\Delta = f_{xx} f_{yy} - f_{xy}^2.$$

$\Delta > 0$	$f_{xx} < 0$ or $f_{yy} < 0$ local maximum $f_{xx} > 0$ or $f_{yy} > 0$ local minimum
$\Delta < 0$	saddle
$\Delta = 0$	more investigation

Example

Find any local maxima, minima or saddles when $z = x^2 + y^2$.

$$z_x = 2x \qquad z_y = 2y.$$

and

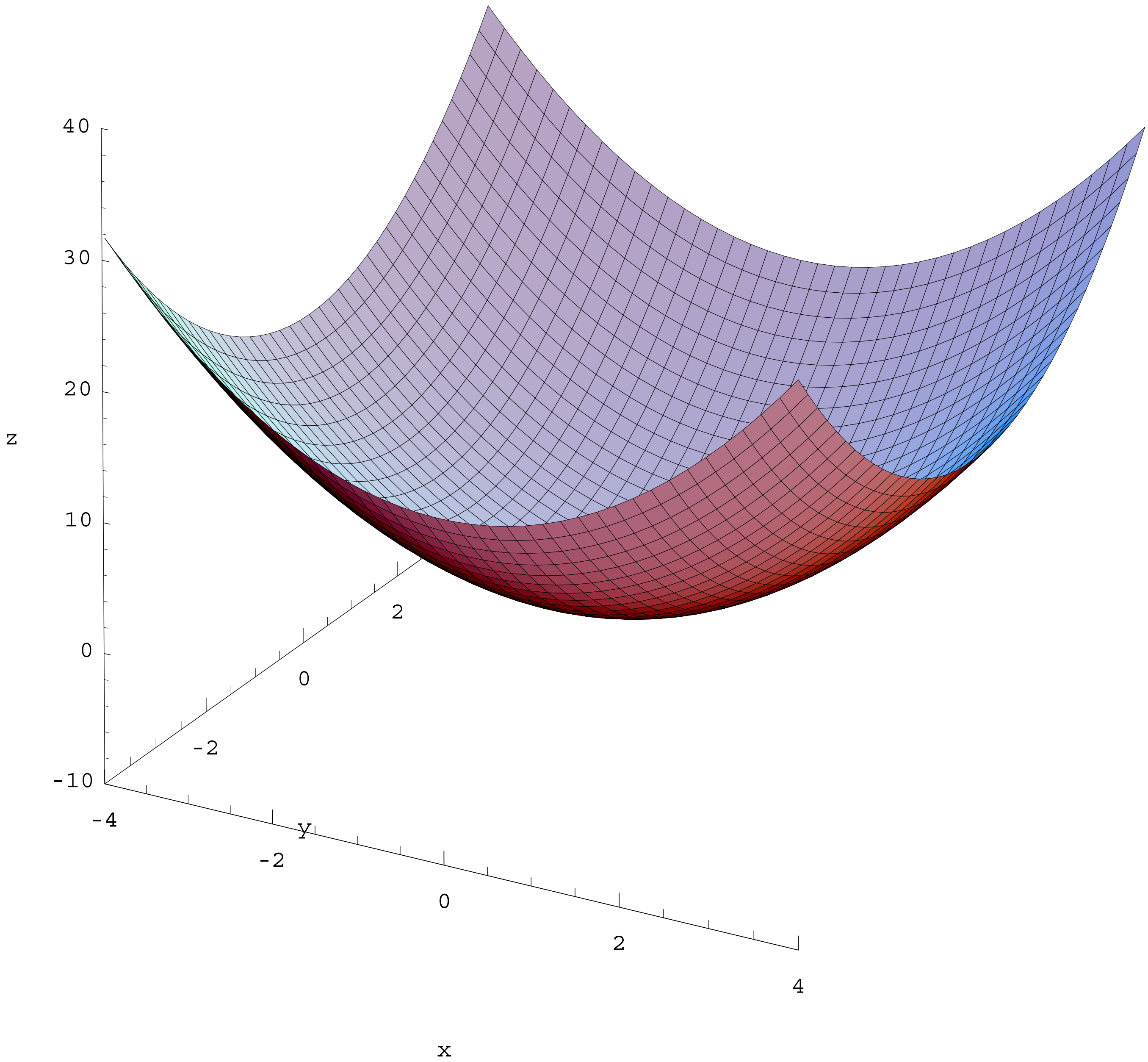
$$z_{xx} = 2 \qquad z_{xy} = 0 \qquad z_{yy} = 2.$$

So $z_x = z_y = 0$ at the origin (only) and, everywhere,

$$\Delta = z_{xx}z_{yy} - z_{xy}^2 = 4 > 0$$

Since $z_{xx} = 2 > 0$, there is a local minimum at the origin.

$$f(x, y) = x^2 + y^2$$



Example

Find any local maxima, minima or saddles when $z = 9 - x^2 - y^2$.

$$z_x = -2x \qquad z_y = -2y.$$

and

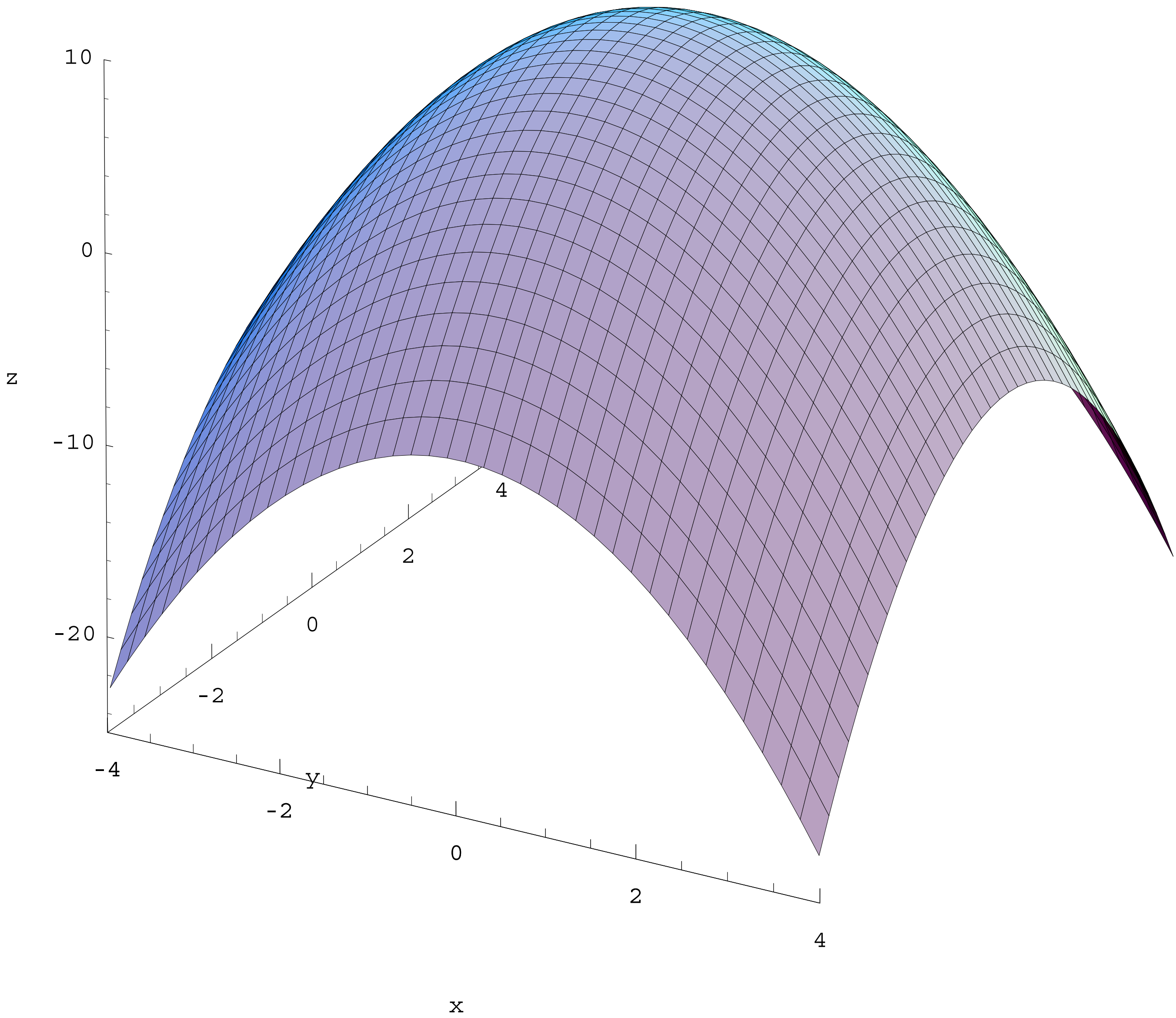
$$z_{xx} = -2 \qquad z_{xy} = 0 \qquad z_{yy} = -2.$$

So $z_x = z_y = 0$ at the origin (only) and, everywhere,

$$\Delta = z_{xx}z_{yy} - z_{xy}^2 = 4 > 0$$

Since $z_{xx} = -2 > 0$, there is a local maximum at the origin.

$$f(x, y) = 9 - x^2 - y^2$$



Example

Find any local maxima, minima or saddles when $z = x^2 - y^2$.

$$z_x = 2x \qquad z_y = -2y.$$

and

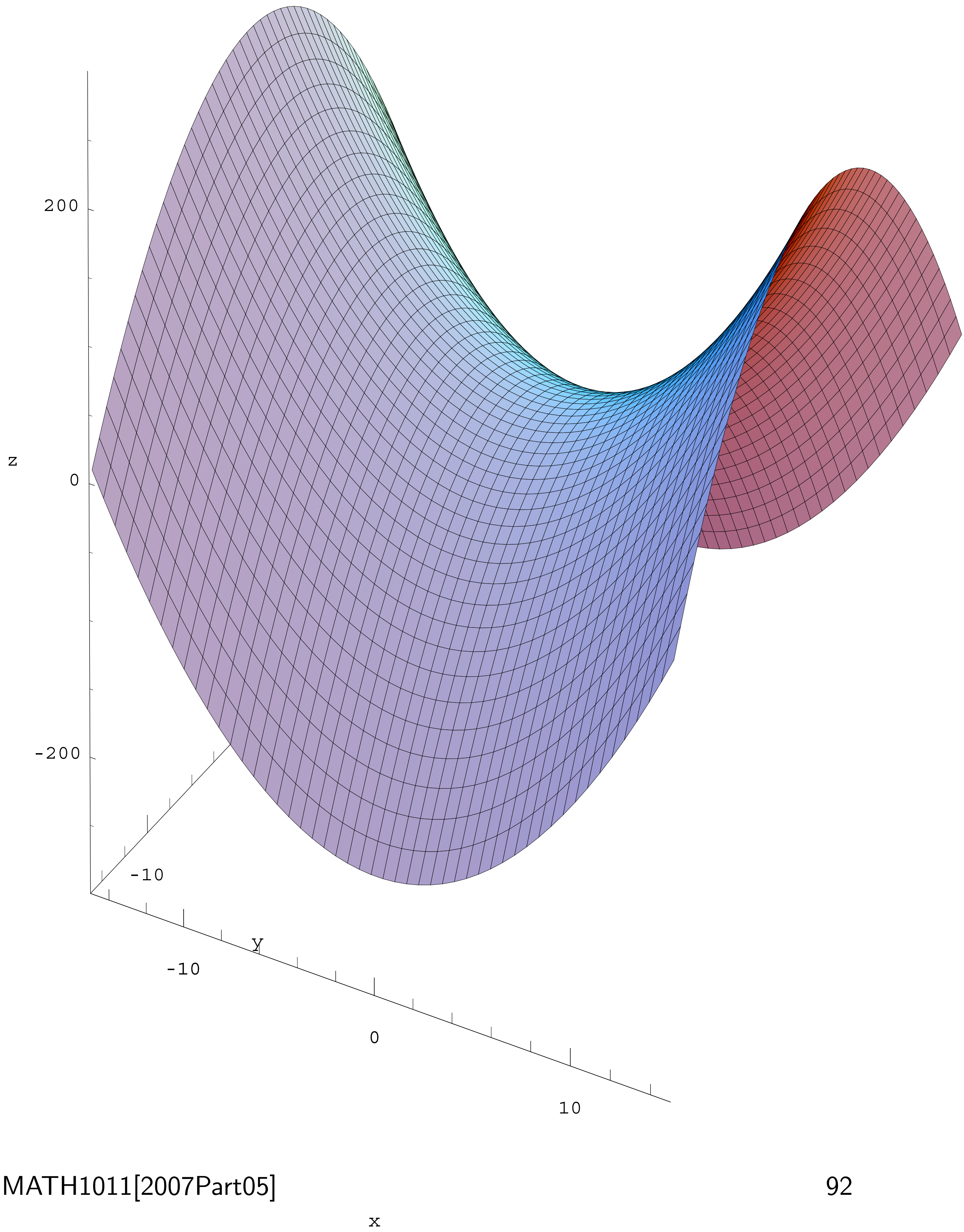
$$z_{xx} = 2 \qquad z_{xy} = 0 \qquad z_{yy} = -2.$$

So $z_x = z_y = 0$ at the origin (only) and, everywhere,

$$\Delta = z_{xx}z_{yy} - z_{xy}^2 = -4 < 0$$

So there is a saddle at the origin.

$$f(x, y) = x^2 - y^2$$



Example

Let $z = x^4 + x^2y^2 + y^4$. Find the tangent plane at $(1, 1, 3)$. Find any local maxima, minima or saddles.

$$z_x = 4x^3+2xy^2 \qquad z_y = 2x^2y+4y^3.$$

Thus

$$z_x(1, 1) = 6 \qquad z_y(1, 1) = 6.$$

So the equation of the tangent plane is

$$z - 3 = 6(x - 1) + 6(y - 1)$$

ie

$$6x + 6y - z = 9.$$

Now

$$z_{xx} = 12x^2 + 2y^2 \quad z_{xy} = 4xy \quad z_{yy} = 2x^2 + 12y^2.$$

Since

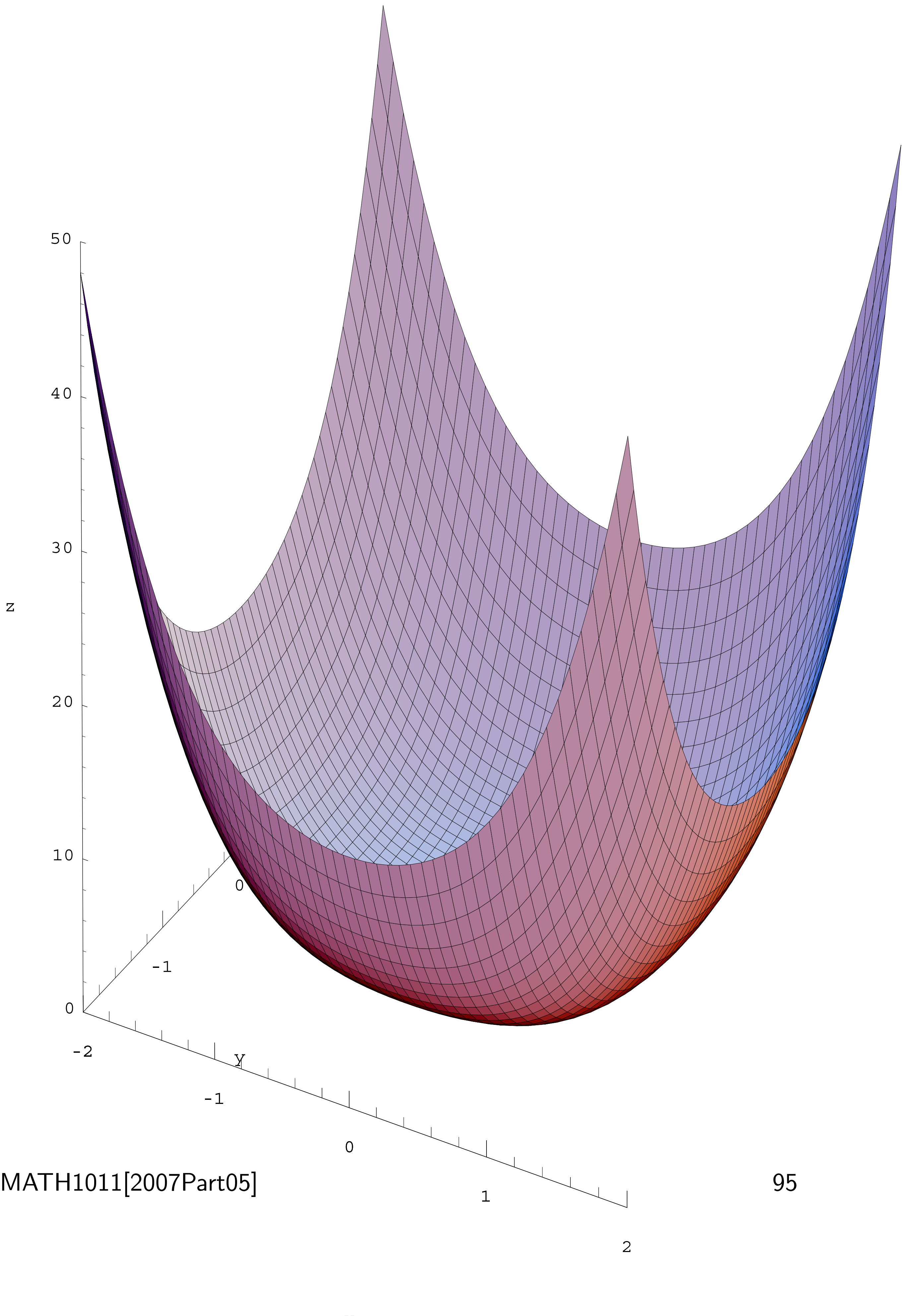
$$z_x = x(4x^2 + 2y^2) \qquad z_y = y(2x^2 + 4y^2),$$

$z_x = z_y = 0$ at the origin (only) and at the origin

$$\Delta = z_{xx}(0,0) z_{yy}(0,0) - z_{xy}(0,0)^2 = 0$$

So our rules do not tell us the nature of the surface near the origin. In fact it is a local minimum.

$$f(x,y) = x^4 + x^2y^2 + y^4$$



Example

Let

$$z = x^3 + 3x^2y + 3xy^2 + y^3 + y^2 - x - y.$$

Find any local maxima, minima or saddles.

$$z_x = 3x^2 + 6xy + 3y^2 - 1$$
$$z_y = 3x^2 + 6xy + 3y^2 + 2y - 1.$$

Setting $f_x = f_y = 0$, we get

$$3(x^2 + 2xy + y^2) - 1 = 0$$
$$3(x^2 + 2xy + y^2) + 2y - 1 = 0.$$

i.e.

$$\begin{aligned} 3(x+y)^2 &= 1 \\ 3(x+y)^2 &= 1-2y. \end{aligned}$$

So $1-2y=1$ and $y=0$. It follows that $3(x+0)^2=1$ and so there are exactly two points where $f_x=f_y=0$: $(\frac{1}{3}\sqrt{3},0)$ and $(-\frac{1}{3}\sqrt{3},0)$.

$$\begin{aligned} z_{xx} &= 6x+6y \\ z_{yy} &= 6x+6y+2 \\ z_{xy} &= 6x+6y \end{aligned}$$

At $(\frac{1}{3}\sqrt{3}, 0)$.

$$\begin{aligned} z_{xx} &= 2\sqrt{3} \\ z_{yy} &= 2\sqrt{3} + 2 \\ z_{xy} &= 2\sqrt{3} \end{aligned}$$

and

$$\Delta = z_{xx}z_{yy} - z_{xy}^2 = 4\sqrt{3} > 0$$

So, since $z_{xx} > 0$ our rules tell us that we have a local minimum at $(\frac{1}{3}\sqrt{3}, 0)$.

At $(-\frac{1}{3}\sqrt{3}, 0)$.

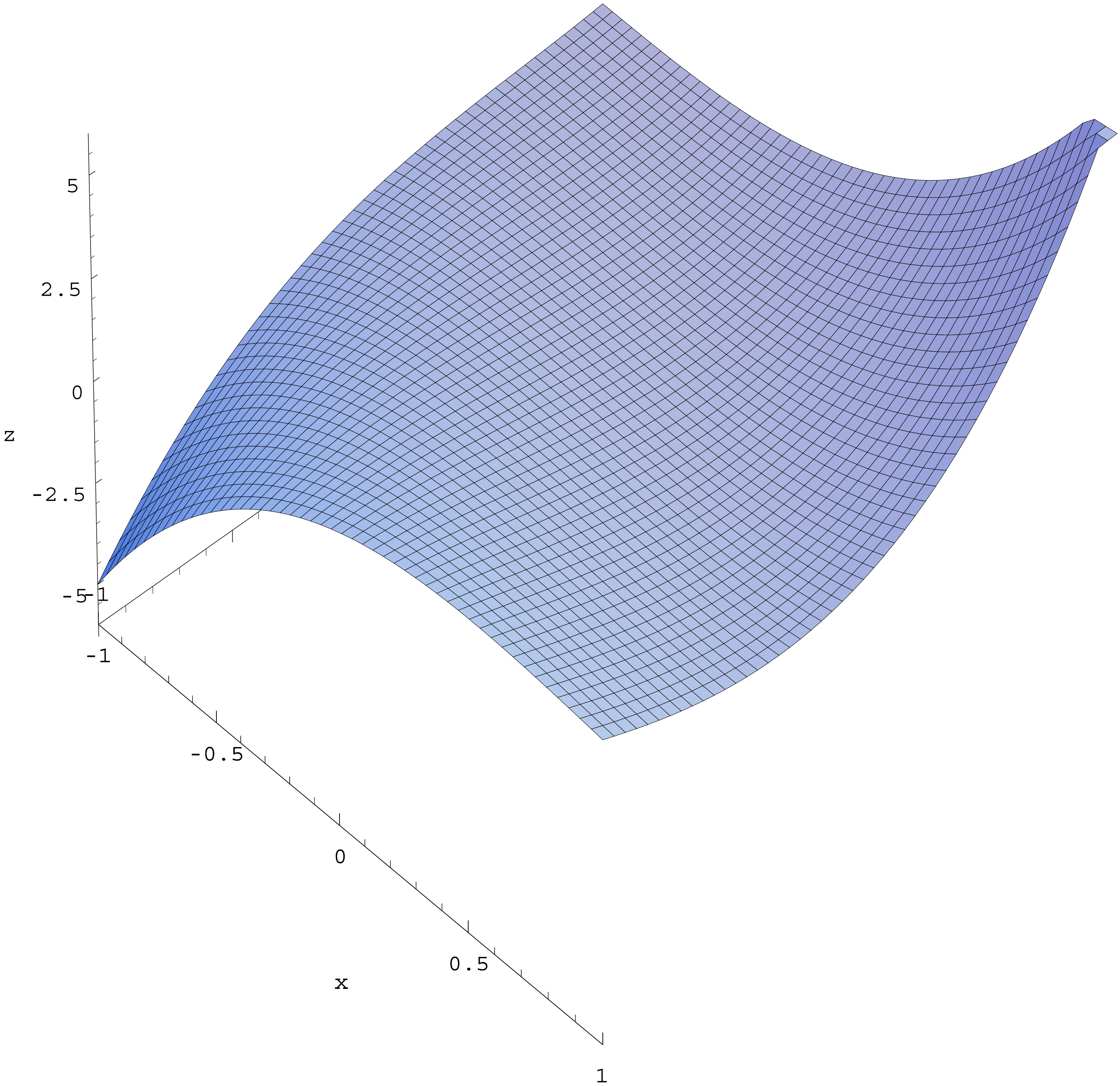
$$z_{xx} = -2\sqrt{3}$$
$$z_{yy} = -2\sqrt{3} + 2$$
$$z_{xy} = -2\sqrt{3}$$

and

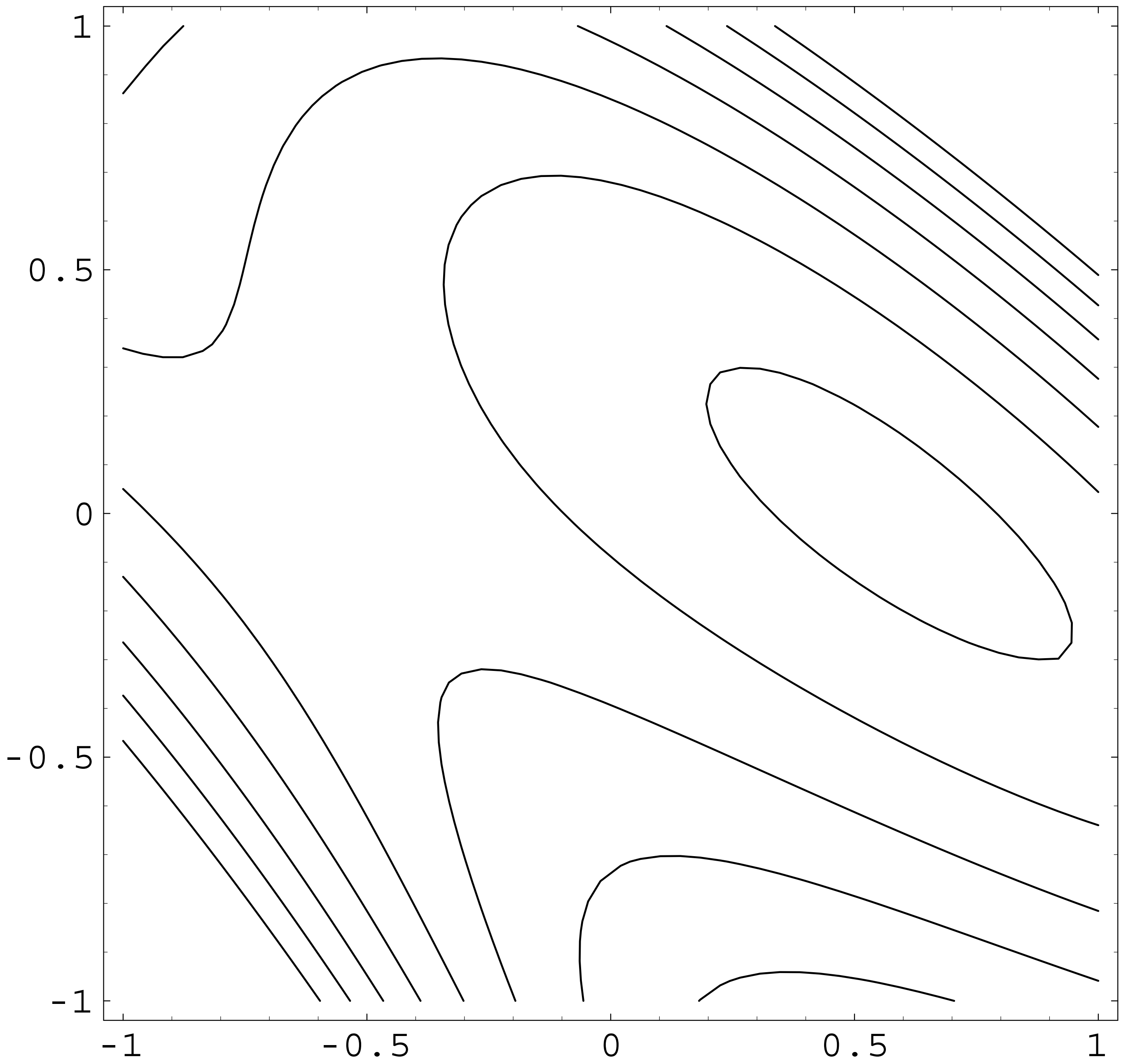
$$\Delta = z_{xx} z_{yy} - z_{xy}^2 = -4\sqrt{3} < 0$$

So, since $z_{xx} > 0$ our rules tell us that we have a saddle at $(-\frac{1}{3}\sqrt{3}, 0)$.

$$f(x,y) = x^3 + 3x^2 + 3xy^2 + y^3 + y^2 - x - y$$



$$f(x,y) = x^3 + 3x^2 + 3xy^2 + y^3 + y^2 - x - y$$

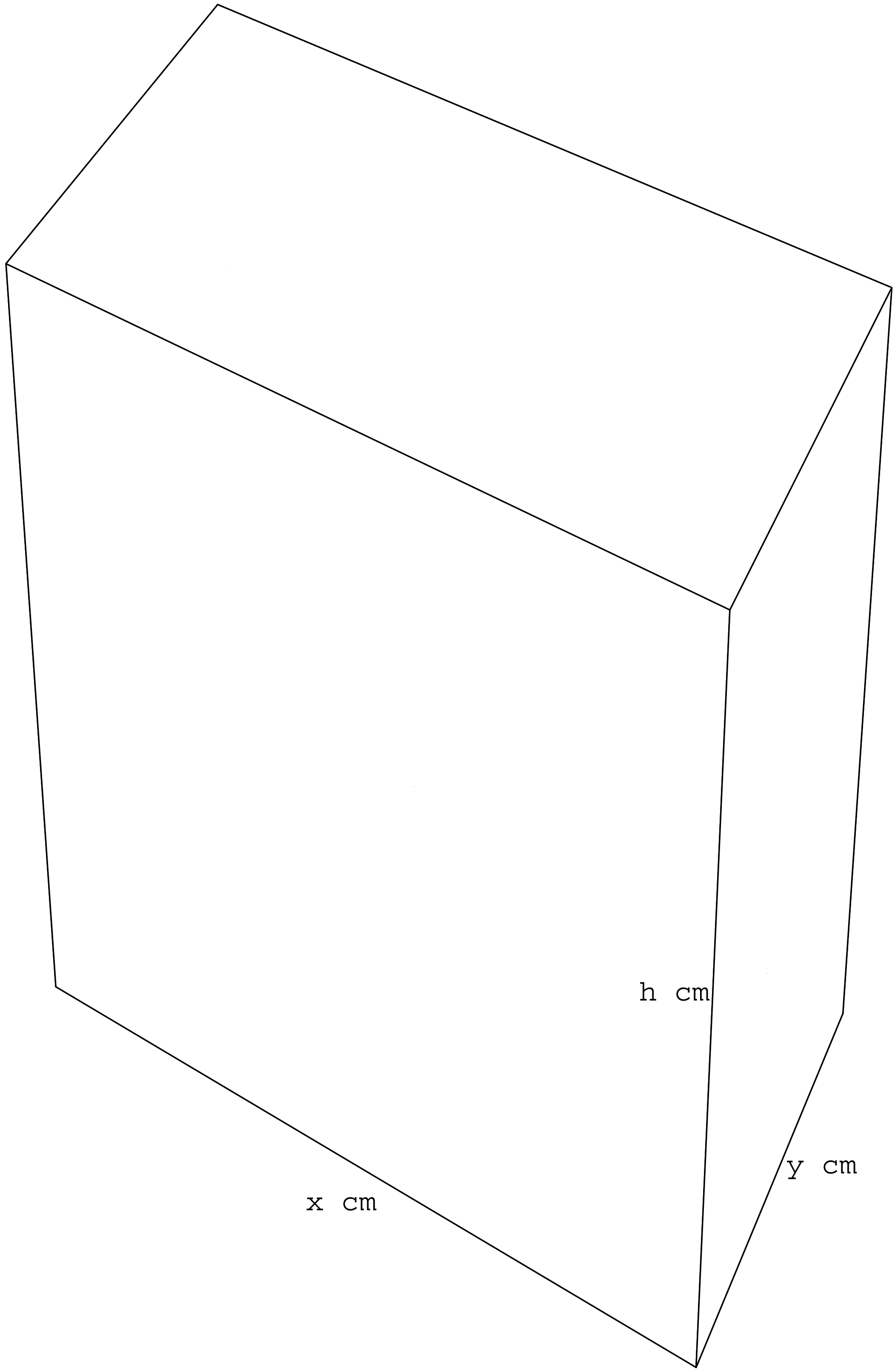


Back to Absolute Maxima and Minima.

Example

A box must have a sum of its length, width and height equal to 1000 cm. Find the shape of box which gives the maximum volume.

Let the length, width and height of the box be x cm, y cm and h cm, respectively. We shall assume that $x \leq y \leq h$.

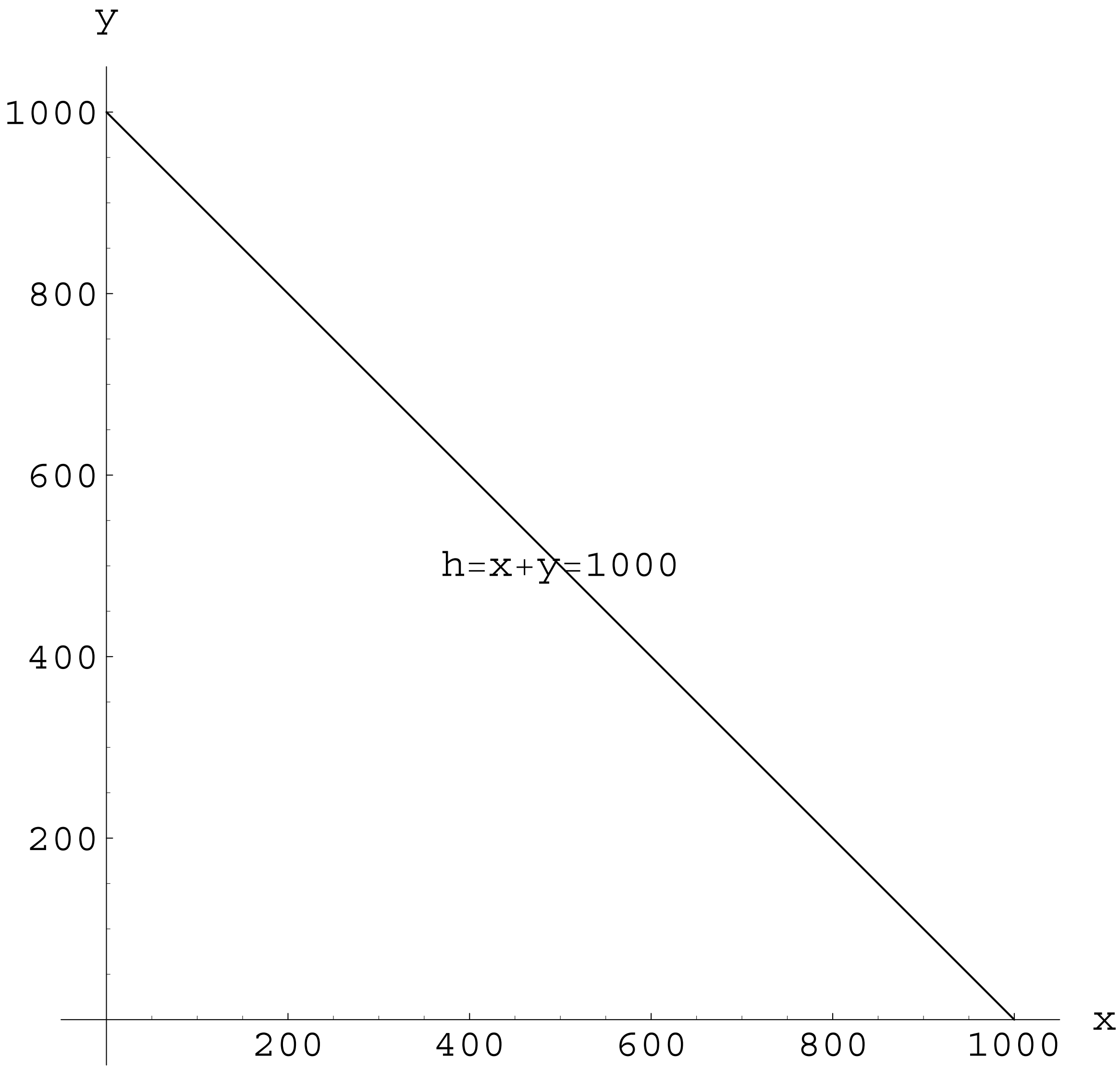

h cm
y cm
x cm

Now $x + y + h = 1000$
so $h = 1000 - x - y$.

If the volume of the box is z cm^3, we have

$$\begin{aligned} z &= hxy \\ &= xy(1000 - x - y) \\ &= 1000xy - x^2y - xy^2 \end{aligned}$$

For practical reasons we shall assume that $x \geq 0$, $y \geq 0$ and $h \geq 0$. This gives the following triangular region of the plane.

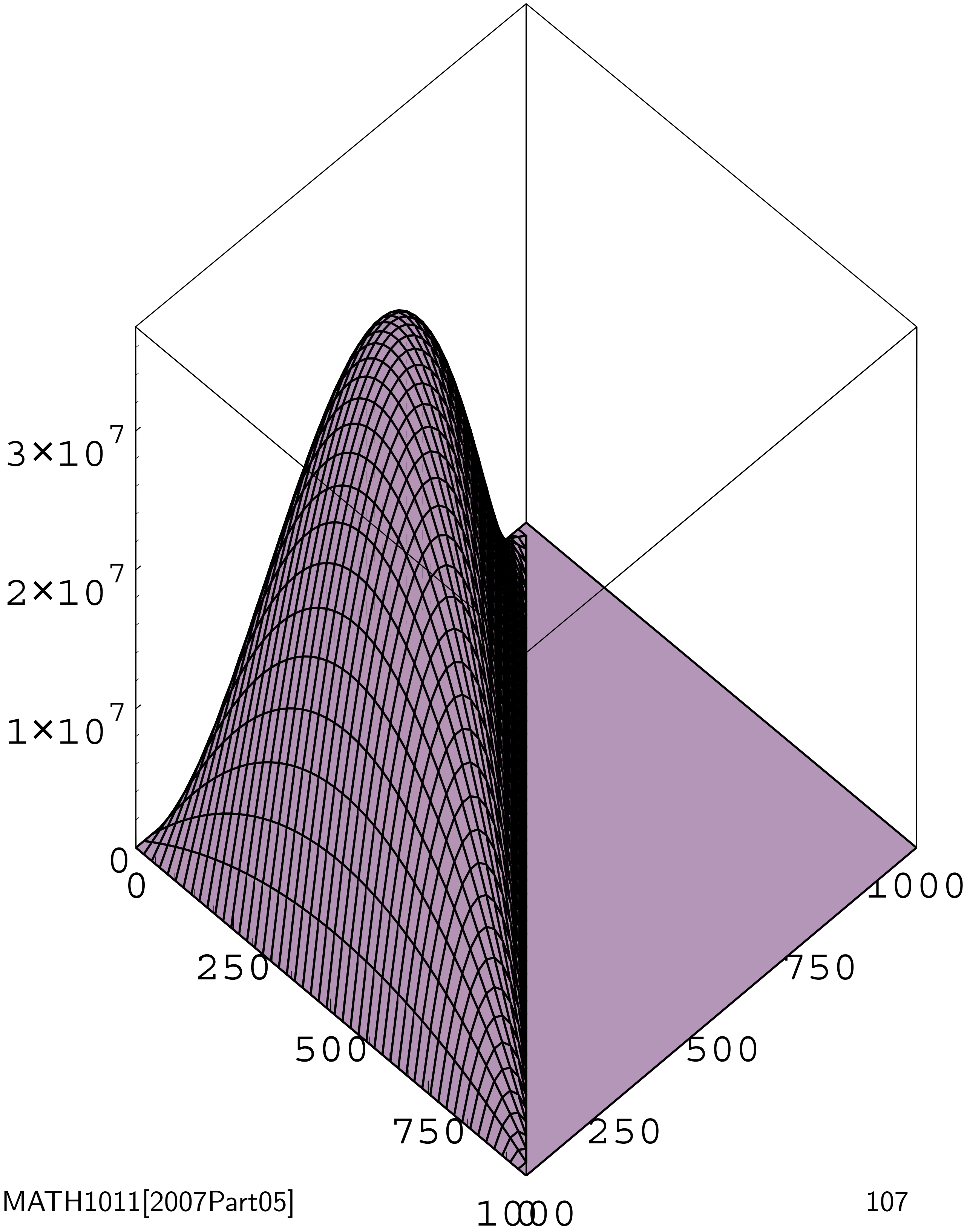


We note that $z = 0$ on the boundary of the triangle and $z > 0$ inside the triangle.

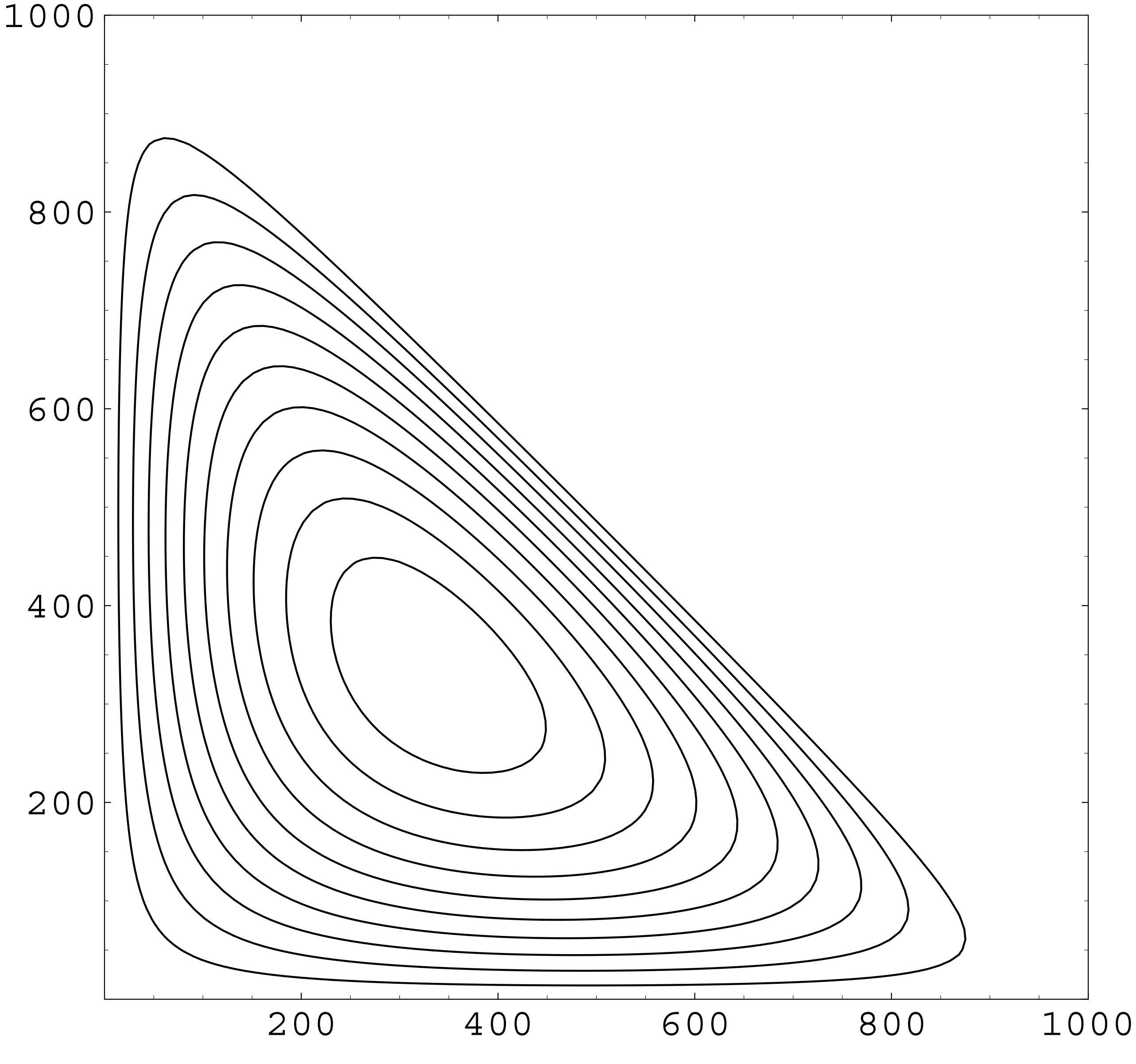
Our problem is to maximise

$$z = 1000xy - x^2y - xy^2$$

over the triangular region will have just described.



3×10^7
2×10^7
1×10^7
0
0
250
500
750
1000
1000
750
500
250
0



1000
800
600
400
200
200
400
600
800
1000

We find the partial derivatives.

$$z_x = 1000y - 2xy - y^2$$
$$z_y = 1000x - x^2 - 2xy$$

Since z_x z_y, are defined everywhere, there are no relevant points of type 2. Since z is positive inside the triangle and zero on the boundary, the boundary points (type 3 points) are irrelevant to finding the maximum.

It follows that there must be a point (or several points) inside the triangle where $z_x = z_y = 0$ and that this point (or one of these points) must give the maximum value of z under the stated conditions.

Setting $z_x = z_y = 0$, we get

$$1000y - 2xy - y^2 = 0$$
$$1000x - x^2 - 2xy = 0$$

i.e.

$$y(2x + y - 1000) = 0$$
$$x(x + 2y - 1000) = 0$$

Since the Maximum is not on the boundary of the triangle, $x \neq 0$ and $y \neq 0$. This leaves

$$2x + y = 1000 = 0$$
$$x + 2y = 1000 = 0$$

Solving the system of linear equations, we get

$$x = \frac{1000}{3}$$
$$y = \frac{1000}{3}$$

So the maximum is attained when

$$x = y = h = \frac{1000}{3}$$

i.e. when the box is a cube of side $\frac{1000}{3}$. The maximum volume is

$$\left(\frac{1000}{3}\right)^3 = \frac{1}{27} \times 10^9 \text{ cm}^3.$$

Example

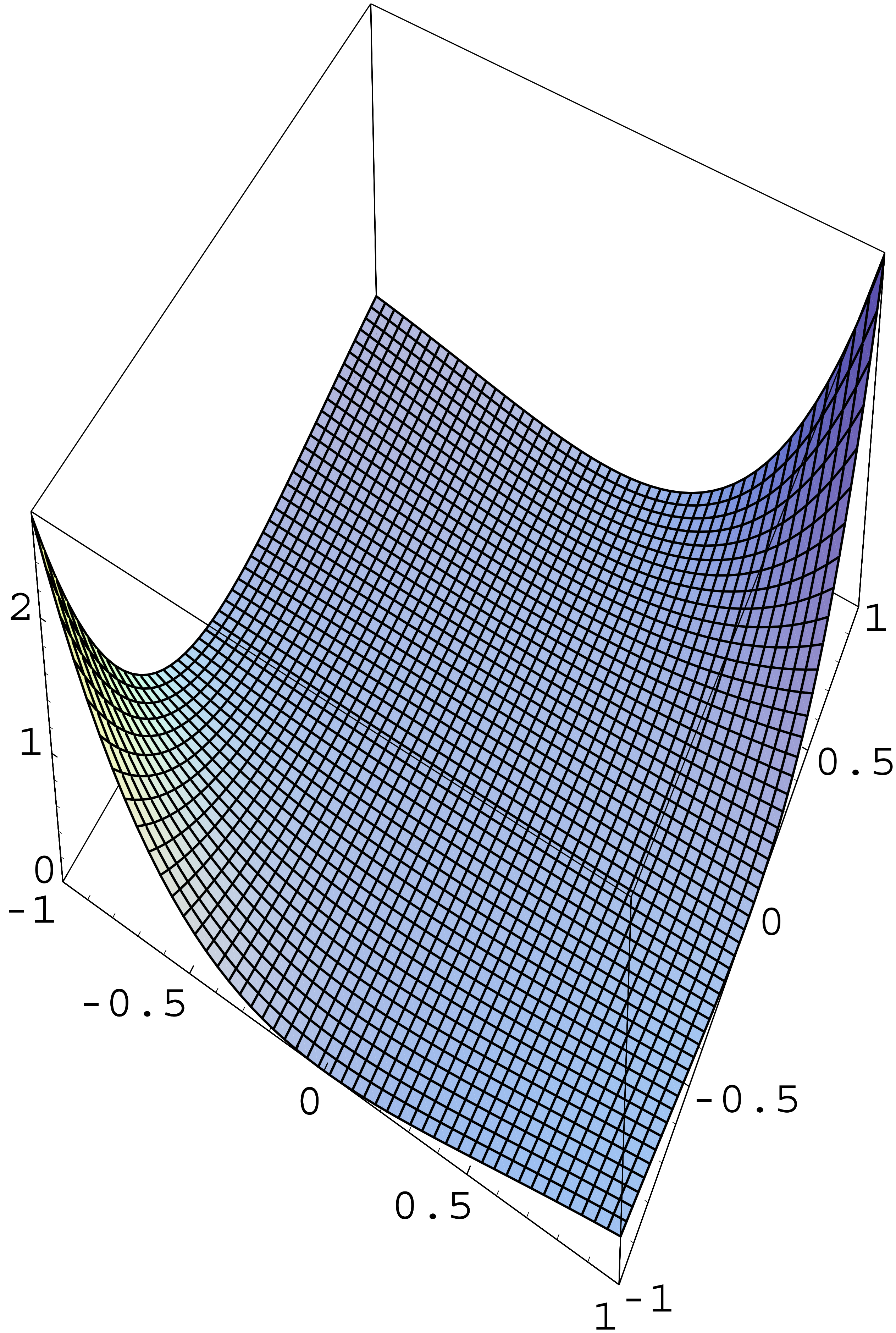
A plate of metal is made in the shape of a square of side 2m. If the plate is thought of as lying in the xy-plane with boundaries $x = \pm 1$ and $y = \pm 1$, the temperature in degrees Celsius at the point (x, y) is

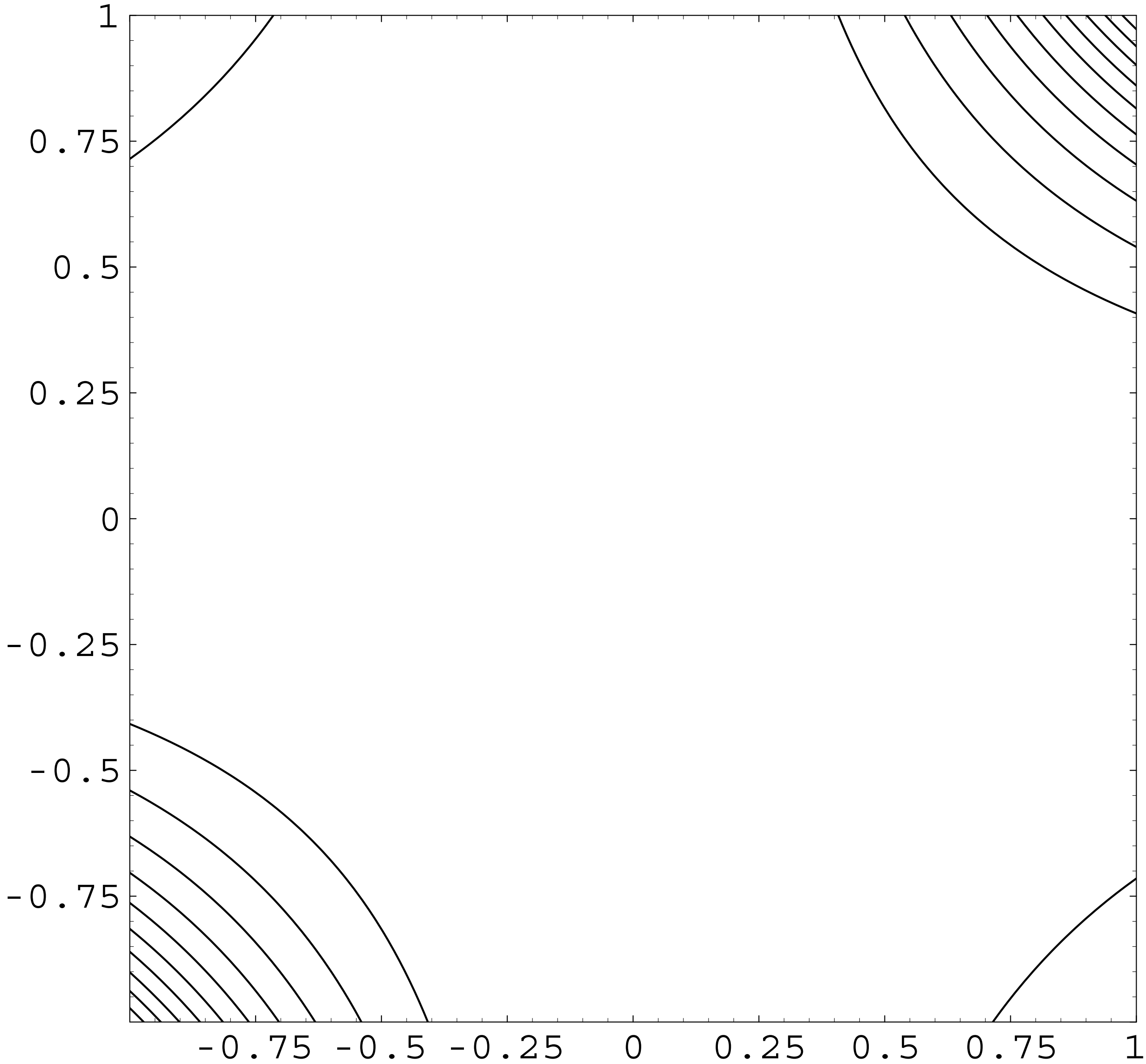
$$T = x^2 y^2 e^{xy}.$$

Find the hottest and coldest points on the plate.

Our problem is to maximise T over the region

$$R = \{(x, y) \mid -1 \le x \le 1, -1 \le y \le 1\}.$$


2
1
0
-1
-0.5
0
0.5
1
-1
-0.5
0
0.5
1


1
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
-0.75
-0.5
-0.25
0
0.25
0.5
0.75
1

We find the partial derivatives.

$$T_x = xy^2\,(2 + xy)\,e^{xy}$$
$$T_y = x^2y\,(2 + xy)\,e^{xy}$$

We see that T_x and T_y are never undefined.

Setting $T_x = T_y = 0$, we get solutions when either $x = 0$ or $y = 0$: the whole of the x-axis and the whole of the y-axis. There are no solutions within the square coming from $2 + xy = 0$.

We have found possible optimal points. We need to check the boundary. This falls naturally into four pieces:

$$B_1 = \{(x, -1) \mid -1 \le x \le 1\}$$
$$B_2 = \{(1, y) \mid -1 \le y \le 1\}$$
$$B_3 = \{(x, 1) \mid -1 \le x \le 1\}$$
$$B_4 = \{(-1, y) \mid -1 \le y \le 1\}.$$

We are left with 4 one variable optimisation problems.

Problem 1

Find the maximum and minimum of $S_1(x) = x^2e^{-x}$ where $-1 \leq x \leq 1$.

Now

$$S_1'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x}.$$

Since $S_1' = 0$ yields $x = 0$ and $x = 2$, the possible maximum and and minimum points are $(-1, -1)$, $(0, -1)$ and $(1, -1)$.

Problem 2

Find the maximum and minimum of $S_2(x) = y^2 e^y$ where $-1 \leq y \leq 1$.

Now

$$S_2'(x) = 2ye^y + y^2 e^y = y(2 + y)e^y.$$

Since $S_2' = 0$ yields $y = 0$ and $x = -2$, the possible maximum and and minimum points are $(1, -1)$, $(1, 0)$ and $(1, 1)$.

Problem 3

Find the maximum and minimum of $S_3(x) = x^2e^x$ where $-1 \leq x \leq 1$.

Now

$$S_3'(x) = 2xe^x + x^2e^x = x(2+x)e^x.$$

Since $S_3' = 0$ yields $x = 0$ and $x = -2$, the possible maximum and and minimum points are $(-1, 1)$, $(0, 1)$ and $(1, 1)$.

Problem 4

Find the maximum and minimum of $S_4(x) = y^2e^{-y}$ where $-1 \leq y \leq 1$.

Now

$$S_4'(x) = 2ye^{-y} - y^2e^{-y} = y(2-y)e^{-y}.$$

Since $S_2' = 0$ yields $y = 0$ and $x = 2$, the possible maximum and and minimum points are $(-1, -1)$, $(-1, 0)$ and $(-1, 1)$.

We have collected the following possibilities for optimal points:

x and y-axes

$(-1,-1)$	$(0,-1)$	$(1,-1)$
$(1,0)$	$(1,1)$	$(0,1)$
$(-1,1)$	$(-1,0)$.	

The corresponding function values are

0		
e	0	e^{-1}
0	e	0
e^{-1}	$0.$	

The minimum temperature is 0°—attained on the x-axis and y-axis. The maximum value is e° attained at $(-1, -1)$ and $(1, 1)$.